On the Performance of Wide-Bandwidth Signal Acquisition in Dense Multipath Channels

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(Invited Paper)

***Abstract*—This paper investigates important properties of acquisition receivers that employ commonly used serial-search strategies. In particular, we focus on the properties of the mean acquisition time (MAT) for wide bandwidth signals in dense multipath channels. We show that a lower bound of the MAT over all possible search strategies is the solution to an integer programming problem with a convex objective function. We also give an upper bound expression for the MAT over all possible search strategies. We demonstrate that the MAT of the fixed-step serial search (FSSS) does not depend on the timing delay of the first resolvable path, thereby simplifying the evaluation of the MAT of the FSSS. The results in this paper can be applied to design and analysis of fast acquisition systems in various wideband scenarios.**

***Index Terms*—Acquisition, dense multipath channels, nonconsecutive serial search, spread-spectrum.**

I. Introduction

WIDE BANDWIDTH transmission systems have emerged as a ubiquitous wireless technology due to their advantages over traditional narrowband systems, and have received considerable attention from the military, commercial, and scientific sectors [1]–[4]. Wide bandwidth transmission systems provide low probability of detection and interception, and allow secure communication in wireless networks. They operate well in extremely challenging environments such as confined, dense urban, and dense multipath areas, where ordinary communication systems may fail to provide reliable transmission. Due primarily to their fine delay resolution properties, wide bandwidth signals are robust against fading and are able to provide accurate positioning. Wide bandwidth systems also allow multiple access communication.

One of the most common forms of wide bandwidth signaling employs spread-spectrum techniques. A spread-spectrum receiver must perform a sequence synchronization, which is required before initiating any communication between the end points. The synchronization process occurs in two stages: the acquisition stage and the tracking stage [5]–[8]. Synchronization time greatly depends on how the receiver performs the acquisition stage, and the acquisition requirement may even limit the capacity of a wireless network [9]. Thus, this paper focuses on the issues related to acquisition.

During the acquisition stage, a receiver performs several tasks. It coarsely aligns the locally generated reference (LGR) sequence with the received signal sequence by testing whether the LGR phase is within the required accuracy of the received signal sequence phase. If not, the receiver will set the new LGR phase according to some prescribed strategy. If the LGR phase is within the required accuracy, the receiver will enter the tracking stage, finely align the two sequences, and maintain the synchronization throughout the communication. In general, the goal of the acquisition system is to minimize the mean acquisition time (MAT); i.e., the average time to achieve the acquisition.

Important parameters associated with the acquisition stage are the total number $N_{\rm unc}$ of phases (cells) to be tested and the number $N_{\rm hit}$ of correct phases (in-phase cells). The expression for $N_{\rm unc}$ is given by

$$N_{\rm unc} = T_{\rm unc}/T_{\rm res} \tag{1}$$

where $T_{\rm unc}$ is the range of the phase delay's uncertainty and $T_{\rm res}$ is the accuracy with which the receiver needs to resolve the phase delay. Without loss of generality, cells are indexed from 1 to $N_{\rm unc}$, and the uncertainty index set

$$\mathcal{U} \stackrel{\triangle}{=} \{1, 2, 3, \ldots, N_{\rm unc}\} \tag{2}$$

denotes a collection of cells to test. Among these $N_{\rm unc}$ cells, $N_{\rm hit}$ cells correspond to the in-phase cells. The quantity $N_{\rm unc}$ is proportional to the number of resolvable paths and is given by

$$N_{\rm hit} = T_{\rm spread}/T_{\rm res} \tag{3}$$

where $T_{\rm spread}$ denotes spread of the multipath dispersion. The set $\mathcal{H}_{\rm hit} \subset \mathcal{U}$ of in-phase cells depends on the timing delays of the resolvable paths, associated with the operating environment.

The set of in-phase cells for a dense multipath channel can be characterized as follows. In such a channel, propagation paths tend to arrive in a cluster [3], [10]–[12]. As a result, if a random variable $B \in \mathcal{U}$ denotes the cell that corresponds to the delay of the first propagation path, the set of in-phase cells conditioned

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on $B = b$ is given by[1]

$$\mathcal{H}_{\text{hit}}(b) \stackrel{\triangle}{=} \{b, b \oplus 1, \ldots, b \oplus (N_{\text{hit}} - 1)\}. \quad (4)$$

Here, the symbol $\oplus$ denotes the modulo N_{unc} addition defined by $x \oplus y \stackrel{\triangle}{=} x + y - lN_{\text{unc}}$ for a unique integer l such that $x + y - lN_{\text{unc}} \in \mathcal{U}$.[2]

For wide bandwidth transmission systems, achieving acquisition in a reasonable amount of time can be a challenging task. In particular, the number N_{unc} of cells that the receiver needs to test can be very large since the quantity $1/T_{\text{res}}$ in (1) is proportional to the transmission bandwidth. This challenge demands a strategy for improving the MAT.

One important approach to improving the MAT is to use an intelligent search order to test cells. A search order can be described by a permutation function π on a set $\mathcal{U}$. The set of all possible search orders is given by

$$\mathcal{P} = \{\pi \mid \pi : \mathcal{U} \to \mathcal{U} \text{ is a permutation function and } \pi(1) = 1\} \quad (5)$$

where the condition $\pi(1) = 1$ simply removes some redundant permutations from $\mathcal{P}$.

The requirement for a search order to be a permutation function, or equivalently a bijection, is important. If the range of a mapping π were not equal to $\mathcal{U}$, the receiver could omit some in-phase cells from the search.[3] To avoid that scenario, we require a search order to be a bijection.

Search orders that have been used in the literature include the conventional serial search (CSS) [13], [14], the fixed-step serial search (FSSS) [15], [16], and the bit-reversal search [16], [17]. In general, search orders affect the MAT, and the notation $\mathbb{E}\{T_{\text{ACQ}}(\pi)\}$ denotes the MAT as a function of a search order π.[4] For a given π, the MAT can be evaluated by using flow diagrams [15], [16], [18]–[20], each corresponding to a different possible position $B = b$ of the first resolvable path. The tuple $(\pi, b) \in \mathcal{P} \times \mathcal{U}$ characterizes the structure of the flow diagram, and we refer to this tuple as a *description*. Note that the flow diagram has one absorption state representing the event of successful acquisition. The average time to arrive at the absorption state is known as the absorption time. This quantity is important and closely related to the MAT.

Although the expression for the MAT, $\mathbb{E}\{T_{\text{ACQ}}(\pi)\}$, can be evaluated for a given π [15], [16], [18]–[20], important properties of the MAT cannot be derived easily. For example, bounds on the minimum MAT, $\min_{\pi \in \mathcal{P}} \mathbb{E}\{T_{\text{ACQ}}(\pi)\}$, are difficult to obtain from the direct optimization over the set of search orders $\mathcal{P}$. The difficulty arises from the fact that the conventional expression of the MAT does not reveal its dependence on the search order π explicitly. To alleviate this difficulty, we propose to transform the set of descriptions into the set of *spacing rules*.[5] It will be apparent that this transformation reveals important properties of the absorption time and enables the investigation of the implications of the absorption time's properties on the MAT. The contributions of this paper are as follows:

- A transformation of a set of descriptions into a set of spacing rules;
- a proof that the absorption time expression in the transform domain is of a quadratic form with a non-negative definite Hessian matrix;
- a proof that the MAT is lower bounded by the solution of an optimization problem, which can be solved algorithmically using well-known methods in convex optimization;
- an explicit upper bound expression for the MAT; and
- a simplification of the MAT expression when the FSSS is employed.

The results here are valid for a broad class of fading conditions, receiver implementations, and operating environments.

This paper is organized as follows. Section II outlines the system model. Section III derives the absorption time expression in a transform domain. Important properties of the absorption time and of the MAT are proved in Sections IV and V, respectively. Section VI concludes the paper and summarizes important findings.

II. System Model

We consider a receiver that employs a widely used serial-search strategy [13]–[16], [18]–[20], [22], [23]. The sequence of phases or cells that the receiver tests during the acquisition stage is given by

$$\pi(k), \pi(k+1), \ldots, \pi(N_{\text{unc}}), \\ \pi(1), \pi(2), \ldots, \pi(N_{\text{unc}}), \pi(1), \pi(2), \ldots \quad (6)$$

where $\pi(k)$ is the first cell that the receiver examines. The subsequence $\{\pi(i)\}_{i=1\ldots N_{\text{unc}}}$ in (6) is repeated to illustrate the fact that, due to noise and fading, the receiver may take several rounds to test the cells before it finds a correct cell.

Search orders that have been used in the literature are shown in Fig. 1. Note from the figure that a search order controls the arrangement of nonabsorbing states in a flow diagram. The CSS [13], [14], where the consecutive cells are tested serially, corresponds to the search order

$$\pi^1(i) = i, \quad 1 \le i \le N_{\text{unc}}. \quad (7)$$

The FSSS [15], [16], which skips $N_{\text{J}} \ge 1$ cells after each test, corresponds to the search order

$$\pi^{N_{\text{J}}}(i) = 1 \oplus (i-1)N_{\text{J}}, \quad 1 \le i \le N_{\text{unc}}. \quad (8)$$

Note that N_{J} and N_{unc} are required to be relatively prime, so that $\pi^{N_{\text{J}}}(\,\cdot\,)$ in (8) is a permutation function and, consequently, a

[1] To emphasize the dependence of $\mathcal{H}_{\text{hit}}$ on $B = b$, we will explicitly write $\mathcal{H}_{\text{hit}}(b)$ as a function of b.

[2] When x and y belongs to $\mathcal{U}$, an integer l is clearly equal to either zero or one.

[3] If all in-phase cells were omitted altogether from the search, the receiver would never acquire the signal.

[4] The acquisition time is a random variable, and the randomness arises from noise, fading, and, possibly, a randomized decision rule at the detection layer.

[5] Our approach follows the general philosophy of solving difficult problems in the transform domains [21].

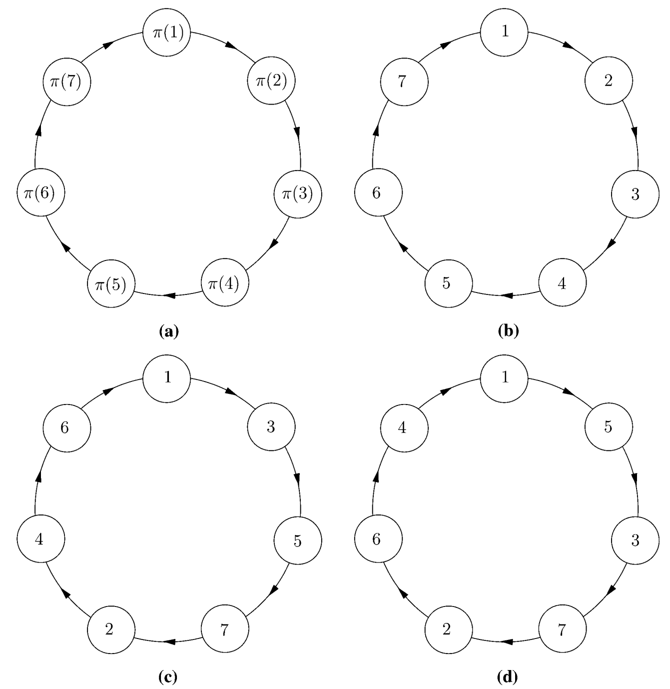

Fig. 1. A receiver tests the cells according to the search order: (a) a generic search order π; (b) the search order π^1 of the CSS; (c) the search order π^2 of the FSSS with the step size $N_{\mathrm{J}} = 2$; (d) the search order π_{R} of the bit-reversal serial search.

member of $\mathcal{P}$. Clearly, the CSS π^1 is a special case of the FSSS $\pi^{N_{\mathrm{J}}}$ with the step size $N_{\mathrm{J}} = 1$. The bit-reversal serial search [16], [17], where the receiver tests the cells in a random-like order, corresponds to the search order π_{R}, defined as follows.[6] For $1 \leq i \neq j \leq N_{\mathrm{unc}}$

$$\pi_{\mathrm{R}}(i) < \pi_{\mathrm{R}}(j) \Leftrightarrow \mathrm{rev}(i) < \mathrm{rev}(j) \tag{9}$$

where $\mathrm{rev}(i)$ is the reversal of the $\lceil \log_2 N_{\mathrm{unc}} \rceil$ binary digit representation of the integer $i - 1$. Equation (9) specifies the unique order of N_{unc} cells in the uncertainty index set: assigning the cost $\mathrm{rev}(i)$ to cell i and arranging the cells in ascending order according to their costs.

A flow diagram represents the details of the acquisition stage, such as the set $\mathcal{H}_{\mathrm{hit}}(\cdot)$ of correct cells, the search order being employed, and the durations and the probabilities associated with the signal detection procedure. Fig. 2 depicts a flow diagram with a generic search order π. The important details of the flow diagram are as follows. The flow diagram contains $N_{\mathrm{unc}} + 1$ states: one absorbing state, N_{hit} states of type H_1, and $N_{\mathrm{unc}} - N_{\mathrm{hit}}$ states of type H_0. The absorbing state, labeled ACQ in the figure, represents the event of successful acquisition. Each H_1-type state corresponds to an in-phase cell, while each H_0-type state corresponds to a non-in-phase cell. Conditioned on $B = b$, the set of H_1 states in (4) can be written in terms of π as

$$\{\pi(k_1), \pi(k_2), \ldots, \pi(k_{N_{\mathrm{hit}}})\} = \mathcal{H}_{\mathrm{hit}}(b) \tag{10}$$

[6] The definition of a bit-reversal search in this paper is a generalization of the definition of a bit-reversal search in [16]. In [16], the authors consider cases when N_{unc} is a power of 2. Here, we allow N_{unc} to be arbitrary.

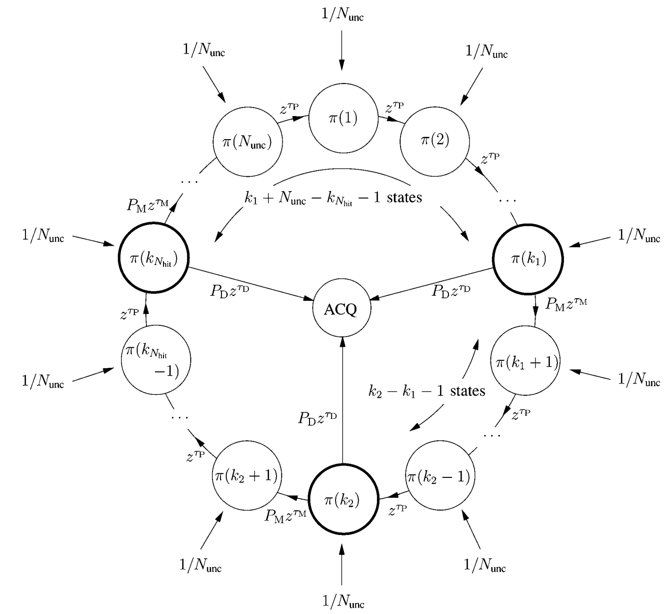

Fig. 2. A generic flow diagram for the serial search with an arbitrary search order π contains $N_{\mathrm{unc}} + 1$ states. The state labeled ACQ is the absorbing state. The states in thick circles are H_1-states, corresponding to the in-phase cells. The remaining states are H_0-states, corresponding to the non-in-phase cells.

for some unique integers $1 \leq k_1 < k_2 < \cdots < k_{N_{\mathrm{hit}}} \leq N_{\mathrm{unc}}$. In a flow diagram, those H_1-states $\pi(k_i)$ have transition paths to the absorbing state.

The probability that the receiver begins a test at any cell is equal to $1/N_{\mathrm{unc}}$. This uniform probability indicates that the receiver has no *a priori* knowledge of the location of a correct cell. The path gain parameters $P_{\mathrm{D}}, P_{\mathrm{M}} \triangleq 1 - P_{\mathrm{D}}, \tau_{\mathrm{D}}, \tau_{\mathrm{M}}$, and τ_{P} are effective probabilities and durations associated with the signal detection procedure.[7] For the purpose of MAT calculation, these parameters can be derived from a generic path gain $H_{\mathrm{D}}(z)$ from an H_1-state to ACQ, a generic path gain $H_{\mathrm{M}}(z)$ from an H_1-state to the adjacent non-absorbing state, and a generic path gain $H_0(z)$ from an H_0-state to the adjacent non-absorbing state [23]. In turn, the details of the signal detection procedure determine $H_{\mathrm{D}}(z)$, $H_{\mathrm{M}}(z)$, and $H_0(z)$ [18]–[20]. With all these features, a flow diagram is a single-absorbing-state Markov chain, having transition probabilities and transition times written in polynomial forms.

The model under consideration is suitable when the power dispersion profile (PDP) is slowly decaying or constant over an interval. Indeed, constant PDPs have been used to study various aspects of spread spectrum systems [24]–[29]. Propagation measurements in urban and suburban environments [30]–[32] and mountainous terrain [33] support such a PDP, since they show that these channels spread the energy over a continuum of arrival times. Thus, this paper employs a basic model to analyze the performance of an acquisition system operating in dense multipath environments.

[7] These parameters are transformations of quantities, such as probabilities of detections, probabilities of false alarms, dwell times, and penalty times.

Flow diagrams are used to find the absorption times and the MAT. The MAT is given by

$$\mathbb{E}\{T_{\rm ACQ}(\pi)\} = \sum_{b=1}^{N_{\rm unc}} \underbrace{\mathbb{E}\{T_{\rm ACQ}(\pi)\,|\,B=b\}}_{\triangleq f(\pi,b)} \cdot \Pr\{B=b\}$$

$$= \sum_{b=1}^{N_{\rm unc}} f(\pi,b)\cdot \Pr\{B=b\} \qquad (11)$$

where $f(\pi,b)$ denotes the absorption time of the flow diagram with a search order π and the set $\mathcal{H}_{\rm hit}(b)$ of in-phase cells. The expression in (11) indicates that the MAT is a convex combination of the absorption times, where the weights are simply the probabilities $\Pr\{B=b\}$. A conventional approach for finding the absorption time yields the absorption time expression

$$f(\pi,b) = \frac{1}{N_{\rm unc}}\frac{d}{dz} \times \left.\left(\frac{\sum_{k=1}^{N_{\rm unc}}\sum_{i=0}^{N_{\rm unc}-1} H^b_{\pi(i\oplus k)}(z)\prod_{j=0}^{i-1} H^b_{\pi(j\oplus k)}(z)}{1-\prod_{i=1}^{N_{\rm unc}} G_i^b(z)}\right)\right|_{z=1} \qquad (12)$$

where the polynomials $H_i^b(z)$ and $G_i^b(z)$ depend on the path gains and equal

$$H_i^b(z) = \begin{cases} P_{\rm D} z^{\tau_{\rm D}} & i\in\mathcal{H}_{\rm hit}(b) \\ 0 & \text{otherwise,}\end{cases}$$

and

$$G_i^b(z) = \begin{cases} P_{\rm M} z^{\tau_{\rm M}} & i\in\mathcal{H}_{\rm hit}(b) \\ z^{\tau_{\rm P}} & \text{otherwise.}\end{cases}$$

Equation (12) follows from a loop-reduction technique, which has been used to find the MATs in [15], [16], [18]–[20].

Although (11) and (12) are suitable for finding the MAT for a *given* search order, they are not suitable for deriving some important properties of the MAT. Note that the expression (12) does not reveal how the absorption time depends on π explicitly. Thus, it is unclear how one can derive or bound the minimum MAT, $\min_{\pi\in\mathcal{P}}\mathbb{E}\{T_{\rm ACQ}(\pi)\}$, and the maximum MAT, $\max_{\pi\in\mathcal{P}}\mathbb{E}\{T_{\rm ACQ}(\pi)\}$, using (11) and (12). To alleviate this difficulty, we transform the set of descriptions into a set of *spacing rules*, following the general philosophy of solving difficult problems in the transform domains [21]. It will be apparent in the following sections that the transformation provides us with important properties of the MAT, $\mathbb{E}\{T_{\rm ACQ}(\pi)\}$.

A spacing rule is an element of the set,[8] shown in (13) at the bottom of the page. A flow diagram with a spacing rule

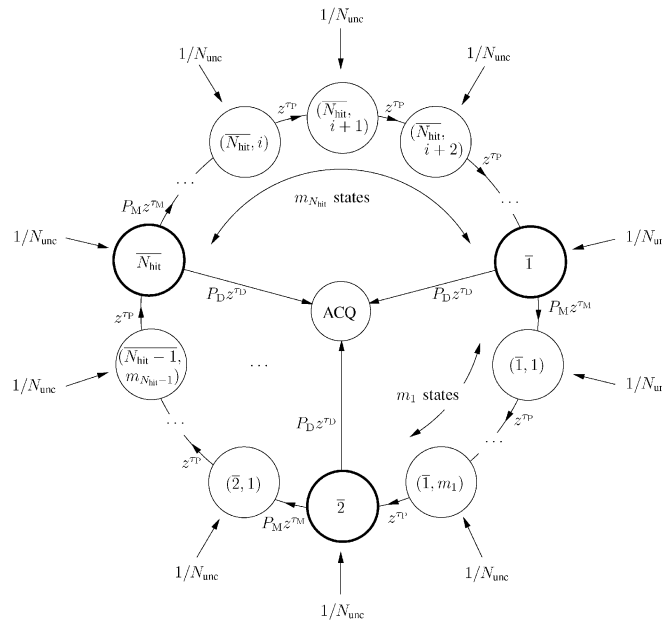

Fig. 3. Spacing rule $\mathbf{m}=[m_1\ m_2\ \cdots\ m_{N_{\rm hit}}]^T$ characterizes the structure of the flow diagram.

$\mathbf{m}\triangleq[m_1\ m_2\ \cdots\ m_{N_{\rm hit}}]^T$ has an H_1-state, which is followed by m_1 H_0-states, which are followed by another H_1-state, which is followed by m_2 H_0-states, and so on (see Fig. 3). Clearly, the sum $\sum_{i=1}^{N_{\rm hit}} m_i$ must equal the number $N_{\rm unc}-N_{\rm hit}$ of H_0-states. Like a description (π,b), a spacing rule characterizes the structure of a flow diagram and strongly affects the absorption time.

A relationship between a description (π,b) and the spacing rule exists. In particular, a flow diagram with a search order π and the set $\mathcal{H}_{\rm hit}(b)$ of in-phase cells has the spacing rule $[m_1\ m_2\ \cdots\ m_{N_{\rm hit}}]^T$, defined as follows:

$$m_i \triangleq \begin{cases} k_{i+1}-k_i-1 & i=1,2,\ldots,N_{\rm hit}-1 \\ k_1+N_{\rm unc}-k_{N_{\rm hit}}-1 & i=N_{\rm hit}\end{cases} \qquad (14)$$

for the unique integers $1\le k_1<k_2<\cdots<k_{N_{\rm hit}}\le N_{\rm unc}$ satisfying (10). A mapping from a description (π,b) to the corresponding spacing rule is denoted by $\mathbf{s}:\mathcal{P}\times\mathcal{U}\to\mathcal{S}$.

Transforming a description (π,b) into a spacing rule provides us with some important properties of the MAT. For example, we will see in Section V that if $v(\,\cdot\,)$ denotes the absorption time as a function of a spacing rule, the MAT will be lower bounded and upper bounded respectively by the integer programming problems $\min_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$ and $\max_{\mathbf{m}\in\mathcal{S}} v(\mathbf{m})$. There are well-known techniques to solve such problems [34]–[36]. In Section III, we derive the explicit expression of $v(\,\cdot\,)$.

[8]The symbol $\mathbb{N}$ denotes a set $\{0,1,2,\ldots\}$ of natural numbers.

$$\mathcal{S} = \left\{[m_1\quad m_2\quad\cdots\quad m_{N_{\rm hit}}]^T \,\middle|\, \sum_{i=1}^{N_{\rm hit}} m_i = N_{\rm unc}-N_{\rm hit};\forall i, m_i\in\mathbb{N}\right\}. \qquad (13)$$

III. Absorption Time Expression

The goal of this section is to derive the explicit absorption time expression $v(\mathbf{m})$ as a function of $\mathbf{m} \in \mathcal{S}$.

Theorem 1 (Absorption Time): The absorption time of the flow diagram with the spacing rule $\mathbf{m} \in \mathcal{S}$ is given by

$$v(\mathbf{m}) = \frac{1}{2}\mathbf{m}^T \mathbf{H}\mathbf{m} + c \tag{15}$$

where, for $1 \le i, j \le N_{\text{hit}}$,

$$\mathbf{H}_{ij} = \frac{\tau_{\text{P}}}{N_{\text{unc}}\left(1 - P_{\text{M}}^{N_{\text{hit}}}\right)} \left[P_{\text{M}}^{N_{\text{hit}}-|i-j|} + P_{\text{M}}^{|i-j|}\right] \tag{16}$$

and

$$c = \left(1 - \frac{N_{\text{hit}}}{N_{\text{unc}}}\right) \cdot \left(\frac{1+P_{\text{M}}}{1-P_{\text{M}}}\right)\frac{\tau_{\text{P}}}{2} + \frac{P_{\text{M}}}{1-P_{\text{M}}}\tau_{\text{M}} + \tau_{\text{D}} \tag{17}$$

with $0^0 \triangleq 1$.

The details of the proof can be found in [17], [23]. In essence, the proof uses the fact that the flow diagram has one absorption state, and thus finding the absorption time is reduced to solving a system of linear equations.

Note that when $P_{\text{M}} = 1$, i.e., the acquisition receiver misses the correct cell with probability one, the absorption time is unbounded. In the following sections, we will assume that $P_{\text{M}} \in [0, 1)$.

IV. Absorption Time Properties

The goal of Section IV is to prove properties of the absorption time.

Theorem 2 (Convexity): The function $v(\cdot)$ is convex on $\mathbb{R}^{N_{\text{hit}}}$.

Proof: Since the coefficient $\tau_{\text{P}}/(N_{\text{unc}}(1 - P_{\text{M}}^{N_{\text{hit}}}))$ in (16) is positive for $P_{\text{M}} \in [0, 1)$, it is sufficient to prove that the $N_{\text{hit}} \times N_{\text{hit}}$ matrix $\mathbf{A}$, in which the ijth entries are given by

$$\mathbf{A}_{ij} \triangleq \left[P_{\text{M}}^{N_{\text{hit}}-|i-j|} + P_{\text{M}}^{|i-j|}\right],$$

is nonnegative definite. When $P_{\text{M}} = 0$, the matrix $\mathbf{A}$ is an identity matrix since $0^0 \triangleq 1$ (see Theorem 1), and thus $\mathbf{A}$ is nonnegative definite. Therefore, we will consider only P_{M} in the range $0 < P_{\text{M}} < 1$.

The matrix $\mathbf{A}$ can be generated from the kernel

$$K(s,t) = \theta e^{|s-t|} + e^{-|s-t|}, \quad -T \le s, t \le T$$

where $T \triangleq -(N_{\text{hit}}/2 - 1/4)\ln P_{\text{M}}$ and $\theta \triangleq P_{\text{M}}^{N_{\text{hit}}}$. In particular, the ijth entries of $\mathbf{A}$ are given by $\mathbf{A}_{ij} = K(t_i, t_j)$, where

$$t_k \triangleq -\left(\frac{2k - N_{\text{hit}} - 1}{2}\right)\ln P_{\text{M}}, \quad k = 1, 2, 3, \ldots, N_{\text{hit}}.$$

Note that $t_k \in (-T, T)$ is in a valid range. See Fig. 4 for an illustration.

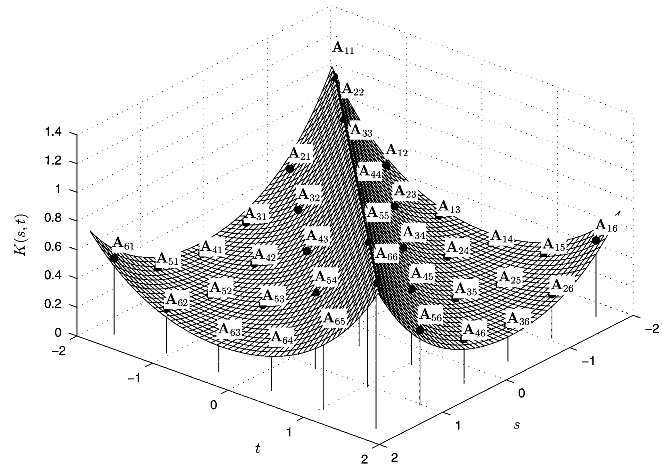

Fig. 4. Entries of matrix $\mathbf{A}$ are generated from the nonnegative definite kernel $K(s,t)$. We show the case when $\mathbf{A}$ is $N_{\text{hit}} \times N_{\text{hit}} = 6 \times 6$.

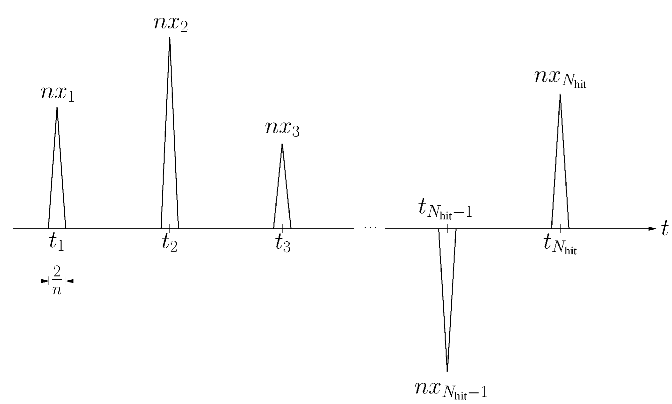

Fig. 5. For $n \ge 1$, the pulse train $x_n(t)$ is constructed from the vector $[x_1\ x_2\ \cdots\ x_{N_{\text{hit}}}]^T$.

Next, we show that $\mathbf{A}$ is nonnegative definite.[9] Assume, to the contrary, that there exists $\mathbf{x} = [x_1\ x_2\ \cdots\ x_{N_{\text{hit}}}]^T \in \mathbb{R}^{N_{\text{hit}}}$ such that $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$. For an integer $n \ge 1$, let

$$p_n(t) = \begin{cases} n^2 t + n & \text{if } t \in \left[-\frac{1}{n}, 0\right] \\ -n^2 t + n & \text{if } t \in \left(0, \frac{1}{n}\right] \\ 0 & \text{otherwise.} \end{cases}$$

Define $x_n(t) \triangleq \sum_{i=1}^{N_{\text{hit}}} x_i p_n(t - t_i)$ (see Fig. 5 for an illustration) and consider a sequence $\{y_n\}$, where

$$y_n \triangleq \int_{-T}^{T}\int_{-T}^{T} x_n(s)K(s,t)x_n(t)\,ds\,dt.$$

Since $\lim_{n\to\infty} y_n = \mathbf{x}^T\mathbf{A}\mathbf{x}$, we can select m large enough so that $y_m < 0$.

It is easy to verify that $T > 0$ and $-e^{-2T} < \theta < e^{-2T}$. Therefore, by Lemma 1 in Appendix I, $K(s,t)$ is nonnegative definite. Note that $x_m(t) \in \mathcal{L}^2$, and, thus, $y_m \ge 0$. Hence, we have a contradiction. That completes the proof. □

[9] Our proof will show the nonnegative definiteness of $\mathbf{A}$ from the nonnegative definiteness of $K(s,t)$. It is interesting to further investigate the positive definiteness of our matrix $\mathbf{A}$.

The next theorem deals with rotational invariance and reversal invariance. Let the rotation and the reversal of any $\mathbf{x} \in \mathbb{R}^{N_{\rm hit}}$ be defined respectively as follows:

$$\mathbf{rot}\{[x_1 \quad x_2 \quad \cdots \quad x_{N_{\rm hit}}]^T\} \stackrel{\triangle}{=} [x_2 \quad x_3 \ \ldots \ x_{N_{\rm hit}} \quad x_1]^T \tag{18}$$

$$\mathbf{rev}\{[x_1 \quad x_2 \quad \cdots \quad x_{N_{\rm hit}}]^T\} \stackrel{\triangle}{=} [x_{N_{\rm hit}} \quad x_{N_{\rm hit}-1} \ \ldots \ x_1]^T. \tag{19}$$

Notice that if $\mathbf{m}$ is a spacing rule, then $\mathbf{rot}\{\mathbf{m}\}$ and $\mathbf{rev}\{\mathbf{m}\}$ are also spacing rules. The next theorem establishes the relationship among the absorption times corresponding to the spacing rules $\mathbf{m}, \mathbf{rot}\{\mathbf{m}\}$, and $\mathbf{rev}\{\mathbf{m}\}$.

Theorem 3 (Rotational Invariance, Reversal Invariance): For every $\mathbf{m} \in \mathcal{S}$,

$$v(\mathbf{m}) = v(\mathbf{rot}\{\mathbf{m}\}) = v(\mathbf{rev}\{\mathbf{m}\}).$$

Proof: For $1 \le i, j \le N_{\rm hit}$, let $h_{i,j}$ denote the ijth entry of matrix $\mathbf{H}$ and $\mathbf{h}_i$ denote the ith column of $\mathbf{H}$. Let any spacing rule $\mathbf{m}$ be given.

To prove the rotational invariance property, we note that we have the first equation at the bottom of the page. The equality (a) follows from the fact that

$$\mathbf{x}^T\mathbf{y} = \mathbf{rot}^T\{\mathbf{x}\}\,\mathbf{rot}\{\mathbf{y}\}, \quad \text{for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^{N_{\rm hit}}.$$

The equality (b) follows from the fact that [10] $h_{i,j} = h_{i \boxplus 1, j \boxplus 1}$ for any $1 \le i, j \le N_{\rm hit}$, which implies that $\mathbf{rot}\{\mathbf{h}_i\} = \mathbf{h}_{i \boxminus 1}$ for any $1 \le i \le N_{\rm hit}$. Therefore, $v(\mathbf{m}) = v(\mathbf{rot}\{\mathbf{m}\})$.

To prove the reversal invariance property, we note that we have the second equation at the bottom of the page. The equality (a) follows from the fact that

$$\mathbf{x}^T\mathbf{y} = \mathbf{rev}^T\{\mathbf{x}\}\,\mathbf{rev}\{\mathbf{y}\}, \quad \text{for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^{N_{\rm hit}}.$$

The equality (b) follows from the fact that $h_{i,j} = h_{(N_{\rm hit}-i+1),(N_{\rm hit}-j+1)}$ for any $1 \le i, j \le N_{\rm hit}$, which implies that $\mathbf{h}_i = \mathbf{rev}\{\mathbf{h}_{N_{\rm hit}-i+1}\}$ for any $1 \le i \le N_{\rm hit}$. Therefore, $v(\mathbf{m}) = v(\mathbf{rev}\{\mathbf{m}\})$. That completes the proof. □

V. MAT Properties

In Section V, we will use the absorption time's properties to derive important properties of the MAT.

Theorem 4 (Lower Bound): The MAT of any search order π satisfies

$$\min_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}) \le \mathbb{E}\{T_{\rm ACQ}(\pi)\}.$$

Proof: Using the expression for the MAT yields

$$\begin{aligned}
\mathbb{E}\{T_{\rm ACQ}(\pi)\} &= \sum_{b=1}^{N_{\rm unc}} f(\pi, b) \cdot \Pr\{B = b\} \\
&= \sum_{b=1}^{N_{\rm unc}} v(\mathbf{s}(\pi, b)) \cdot \Pr\{B = b\} \\
&\ge \sum_{b=1}^{N_{\rm unc}} \min_{(\tilde{\pi}, \tilde{b}) \in \mathcal{P} \times \mathcal{U}} v(\mathbf{s}(\tilde{\pi}, \tilde{b})) \cdot \Pr\{B = b\} \\
&= \sum_{b=1}^{N_{\rm unc}} \min_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}) \cdot \Pr\{B = b\} \\
&= \min_{\mathbf{m} \in \mathcal{S}} v(\mathbf{m}).
\end{aligned}$$

That completes the proof. □

Note that the objective function $v(\,\cdot\,)$ is convex on $\mathbb{R}^{N_{\rm hit}}$ by Theorem 2, and well known techniques for solving

[10] The symbol $\boxplus$ denotes the modulo $N_{\rm hit}$ addition defined by $a \boxplus b \stackrel{\triangle}{=} a + b - lN_{\rm hit}$ for a unique integer l such that $1 \le a + b - lN_{\rm hit} \le N_{\rm hit}$. We write $a \boxminus b$ for $a \boxplus (-b)$.

$$\begin{aligned}
\mathbf{m}^T\mathbf{H}\mathbf{m} &= \mathbf{m}^T \begin{bmatrix} | & | & & | & | \\ \mathbf{h}_2 & \mathbf{h}_3 & \cdots & \mathbf{h}_{N_{\rm hit}} & \mathbf{h}_1 \\ | & | & & | & | \end{bmatrix} \mathbf{rot}\{\mathbf{m}\} \\
&\stackrel{(a)}{=} \mathbf{rot}^T\{\mathbf{m}\} \begin{bmatrix} | & | & & | & | \\ \mathbf{rot}\{\mathbf{h}_2\} & \mathbf{rot}\{\mathbf{h}_3\} & \cdots & \mathbf{rot}\{\mathbf{h}_{N_{\rm hit}}\} & \mathbf{rot}\{\mathbf{h}_1\} \\ | & | & & | & | \end{bmatrix} \mathbf{rot}\{\mathbf{m}\} \\
&\stackrel{(b)}{=} \mathbf{rot}^T\{\mathbf{m}\}\,\mathbf{H}\,\mathbf{rot}\{\mathbf{m}\}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{m}^T\mathbf{H}\mathbf{m} &= \mathbf{m}^T \begin{bmatrix} | & | & & | & | \\ \mathbf{h}_{N_{\rm hit}} & \mathbf{h}_{N_{\rm hit}-1} & \cdots & \mathbf{h}_2 & \mathbf{h}_1 \\ | & | & & | & | \end{bmatrix} \mathbf{rev}\{\mathbf{m}\} \\
&\stackrel{(a)}{=} \mathbf{rev}^T\{\mathbf{m}\} \begin{bmatrix} | & | & & | & | \\ \mathbf{rev}\{\mathbf{h}_{N_{\rm hit}}\} & \mathbf{rev}\{\mathbf{h}_{N_{\rm hit}-1}\} & \cdots & \mathbf{rev}\{\mathbf{h}_2\} & \mathbf{rev}\{\mathbf{h}_1\} \\ | & | & & | & | \end{bmatrix} \mathbf{rev}\{\mathbf{m}\} \\
&\stackrel{(b)}{=} \mathbf{rev}^T\{\mathbf{m}\}\,\mathbf{H}\,\mathbf{rev}\{\mathbf{m}\}.
\end{aligned}$$

integer programming problems with convex objective functions are available [34]–[36].

Theorem 5 (Upper Bound): The MAT of any search order π satisfies

$$\mathbb{E}\{T_{\mathrm{ACQ}}(\pi)\} \leq T_{\max}$$

where

$$\begin{aligned} T_{\max} &\overset{\triangle}{=} v([N_{\mathrm{unc}} - N_{\mathrm{hit}} \quad 0 \quad 0 \quad \ldots \quad 0]^T) \\ &= \frac{(N_{\mathrm{unc}} - N_{\mathrm{hit}})^2}{N_{\mathrm{unc}}} \cdot \left(\frac{1 + P_{\mathrm{M}}^{N_{\mathrm{hit}}}}{1 - P_{\mathrm{M}}^{N_{\mathrm{hit}}}}\right) \frac{\tau_{\mathrm{P}}}{2} \\ &\quad + \left(1 - \frac{N_{\mathrm{hit}}}{N_{\mathrm{unc}}}\right) \cdot \left(\frac{1 + P_{\mathrm{M}}}{1 - P_{\mathrm{M}}}\right) \frac{\tau_{\mathrm{P}}}{2} + \frac{P_{\mathrm{M}}}{1 - P_{\mathrm{M}}} \tau_{\mathrm{M}} + \tau_{\mathrm{D}}. \end{aligned}$$

Proof: For $1 \leq i \leq N_{\mathrm{hit}}$, let $\mathbf{e}_i = [0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0]^T$ denote a standard basis vector in $\mathbb{R}^{N_{\mathrm{hit}}}$ with one and only one nonzero element at the ith component. Let $\mathcal{E} \overset{\triangle}{=} \{(N_{\mathrm{unc}} - N_{\mathrm{hit}})\mathbf{e}_i$, for all $1 \leq i \leq N_{\mathrm{hit}}\}$ denote a subset of $\mathcal{S}$. Clearly, $\mathcal{E}$ forms a basis for $\mathbb{R}^{N_{\mathrm{hit}}}$.

Any spacing rule $\mathbf{m} = [m_1 \ m_2 \ \cdots \ m_{N_{\mathrm{hit}}}]^T \in \mathcal{S} \subset \mathbb{R}^{N_{\mathrm{hit}}}$ can be written as

$$\mathbf{m} = \sum_{i=1}^{N_{\mathrm{hit}}} \lambda_i \cdot [(N_{\mathrm{unc}} - N_{\mathrm{hit}})\mathbf{e}_i] \tag{20}$$

where

$$\lambda_i = \frac{m_i}{N_{\mathrm{unc}} - N_{\mathrm{hit}}} \quad i = 1, 2, \ldots, N_{\mathrm{hit}}. \tag{21}$$

Note that $\lambda_i \geq 0$, for $i = 1, 2, \ldots, N_{\mathrm{hit}}$ and $\sum_{i=1}^{N_{\mathrm{hit}}} \lambda_i = 1$. Thus, $\mathbf{m}$ in (20) is written as a convex combination of the spacing rules in $\mathcal{E}$.

Recall that $v(\cdot)$ is convex on $\mathbb{R}^{N_{\mathrm{hit}}}$ (Theorem 2). Then, we have

$$\begin{aligned} v(\mathbf{m}) &\leq \sum_{i=1}^{N_{\mathrm{hit}}} \lambda_i \cdot v((N_{\mathrm{unc}} - N_{\mathrm{hit}})\mathbf{e}_i \\ &\overset{(a)}{=} \sum_{i=1}^{N_{\mathrm{hit}}} \lambda_i \cdot v((N_{\mathrm{unc}} - N_{\mathrm{hit}})\mathbf{e}_1) \\ &= v((N_{\mathrm{unc}} - N_{\mathrm{hit}})\mathbf{e}_1) \\ &\overset{(b)}{=} T_{\max} \end{aligned}$$

where the equality (a) follows from the rotational invariance property (Theorem 3), and the equality (b) follows directly from the absorption time expression (Theorem 1). That completes the proof. □

The next theorem implies that the MAT of a receiver employing the FSSS does not depend on the location of the first in-phase cell. Thus, the MAT expression can be simplified for the case of FSSS.

Theorem 6 (Simplification): Let $\mathbf{s}(\pi^{N_{\mathrm{J}}}, b)$ denote the spacing rule for the description $(\pi^{N_{\mathrm{J}}}, b)$ corresponding to the FSSS and the location $1 \leq b \leq N_{\mathrm{unc}}$ of the first in-phase cell. Then

$$v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, 1)) = v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, 2)) = \cdots = v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, N_{\mathrm{hit}}))$$

and the MAT expression for the FSSS is equal to

$$\mathbb{E}\{T_{\mathrm{ACQ}}(\pi^{N_{\mathrm{J}}})\} = v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, 1)).$$

Proof: Let the step size N_{J} be given. Conditioned on $B = 1$, the set of in-phase cells is given by

$$\begin{aligned} \mathcal{H}_{\mathrm{hit}}(1) &= \{1, 2, \ldots, N_{\mathrm{hit}}\} \\ &= \{\pi^{N_{\mathrm{J}}}(k_1), \pi^{N_{\mathrm{J}}}(k_2), \ldots, \pi^{N_{\mathrm{J}}}(k_{N_{\mathrm{hit}}})\} \end{aligned} \tag{22}$$

for some unique integers $1 = k_1 < k_2 < \cdots < k_{N_{\mathrm{hit}}}$. We now transform the description $(\pi^{N_{\mathrm{J}}}, 1)$ into the spacing rule $\mathbf{m}$, where the components m_i are given in (14). For any $j \geq 1$, let

$$\mathbf{rot}_j\{\mathbf{x}\} \overset{\triangle}{=} \underbrace{\mathbf{rot}\{\mathbf{rot}\{\cdots\{\mathbf{rot}}_{j\text{times}}\{\mathbf{x}\}\}\cdots\}\}$$

denote a vector obtained from the rotations of $\mathbf{x} \in \mathbb{R}^{N_{\mathrm{hit}}}$ for j times. Let

$$\mathcal{R} \overset{\triangle}{=} \{\mathbf{m}, \mathbf{rot}\{\mathbf{m}\}, \mathbf{rot}_2\{\mathbf{m}\}, \ldots, \mathbf{rot}_{N_{\mathrm{hit}}-1}\{\mathbf{m}\}\}$$

denote a set of all rotations of the spacing rule $\mathbf{m}$. By construction, $\mathbf{s}(\pi^{N_{\mathrm{J}}}, 1) = \mathbf{m} \in \mathcal{R}$.

Let any b with $2 \leq b \leq N_{\mathrm{unc}}$ be given. We want to show that $\mathbf{s}(\pi^{N_{\mathrm{J}}}, b) \in \mathcal{R}$.

Consider a flow diagram with the description $(\pi^{N_{\mathrm{J}}}, b)$. Then, the set of in-phase cells is[11]

$$\begin{aligned} \mathcal{H}_{\mathrm{hit}}(b) &= \mathcal{H}_{\mathrm{hit}}(1) \oplus (b-1) \\ &= \{1 \oplus (k_1 - 1)N_{\mathrm{J}}, 1 \oplus (k_2 - 1)N_{\mathrm{J}}, \ldots, \\ &\quad 1 \oplus (k_{N_{\mathrm{hit}}} - 1)N_{\mathrm{J}}\} \oplus (b-1) \\ &= \{b \oplus (k_1 - 1)N_{\mathrm{J}}, b \oplus (k_2 - 1), \ldots, b \oplus (k_{N_{\mathrm{hit}}} - 1)\} \end{aligned} \tag{23}$$

where the second equality follows from the last equality of (22) and from the definition of the FSSS in (8).

Let $x =^{\triangle} b \oplus (k_1 - 1)N_{\mathrm{J}}$ and $y =^{\triangle} b \oplus (k_2 - 1)N_{\mathrm{J}}$ denote elements of $\mathcal{H}_{\mathrm{hit}}(b)$. Then, x and y are H_1-states. For any $k_1 < j < k_2$, we have $b \oplus (j-1)N_{\mathrm{J}} \notin \mathcal{H}_{\mathrm{hit}}(b)$ since $k_2 < k_3 < \cdots < k_{N_{\mathrm{hit}}}$. Thus, $k_2 - k_1 + 1 = m_1$ states in $\{b \oplus (j - 1)N_{\mathrm{J}} \mid k_1 < j < k_2\}$ are complete H_0-states between the two neighboring H_1-states x and y. A similar argument will show that, for $1 \leq i \leq N_{\mathrm{hit}}$, the quantity m_i in (14) is the number of H_0-states between two neighboring states $b \oplus (k_i - 1)N_{\mathrm{J}}$ and $b \oplus (k_{i \boxplus 1} - 1)N_{\mathrm{J}}$. Therefore,

$$\mathbf{s}(\pi^{N_{\mathrm{J}}}, b) = \mathbf{rot}_l\{\mathbf{m}\} \in \mathcal{R}, \quad \text{for some } l \geq 1.$$

All spacing rules in $\mathcal{R}$ have the same absorption time by the rotational invariance property (Theorem 3). Therefore, $v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, 1)) = v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, 2)) = \cdots = v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, N_{\mathrm{hit}}))$, which implies that the MAT expression for the FSSS is $\mathbb{E}\{T_{\mathrm{ACQ}}(\pi^{N_{\mathrm{J}}})\} = v(\mathbf{s}(\pi^{N_{\mathrm{J}}}, 1))$. That completes the proof. □

The next theorem is an application of the reversal invariance property and the simplification theorem.

[11] For a set $\mathcal{A}$ of integers and a fixed integer n, define $\mathcal{A} \oplus n \overset{\triangle}{=} \{m \oplus n \mid m \in \mathcal{A}\}$.

Theorem 7 (Equivalent Pair): For any step size $N_{\rm J}$, the FSSS's $\pi^{N_{\rm J}}$ and $\pi^{N_{\rm unc}-N_{\rm J}}$ yield an identical MAT: $\mathbb{E}\{T_{\rm ACQ}(\pi^{N_{\rm J}})\} = \mathbb{E}\{T_{\rm ACQ}(\pi^{N_{\rm unc}-N_{\rm J}})\}$.

Proof: The definition of the FSSS indicates that

$$\pi^{N_{\rm J}}(1) = \pi^{N_{\rm unc}-N_{\rm J}}(1)$$
$$\pi^{N_{\rm J}}(k) = \pi^{N_{\rm unc}-N_{\rm J}}(N_{\rm unc}+k-2), \quad 2 \le k \le N_{\rm unc}.$$

In other words, the FSSS $\pi^{N_{\rm J}}$ yields the search sequence, for some cells $\{c_i\}$,

$$\ldots, c_1, c_2, \ldots, c_{N_{\rm unc}-1}, c_{N_{\rm unc}},\ c_1, c_2, \ldots, c_{N_{\rm unc}-1}, c_{N_{\rm unc}},\ \ldots$$

if and only if the FSSS $\pi^{N_{\rm unc}-N_{\rm J}}$ yields the search sequence

$$\ldots, c_1, c_{N_{\rm unc}}, c_{N_{\rm unc}-1}, \ldots, c_2,\ c_1, c_{N_{\rm unc}}, c_{N_{\rm unc}-1}, \ldots, c_2,\ \ldots.$$

Therefore, the spacing rule $\mathbf{s}(\pi^{N_{\rm J}},1)$ that corresponds to the description $(\pi^{N_{\rm J}},1)$ is the reversal of the spacing rule $\mathbf{s}(\pi^{N_{\rm unc}-N_{\rm J}},1)$:

$$\mathbf{rev}\{\mathbf{s}(\pi^{N_{\rm J}},1)\} = \mathbf{s}(\pi^{N_{\rm unc}-N_{\rm J}},1).$$

The theorem's statement then follows from the reversal invariance (Theorem 3) and the simplification theorem (Theorem 6).

VI. Conclusion

This paper investigates the properties of acquisition receivers that employ serial-search strategies. We begin by noting that the mean acquisition time (MAT) is a convex combination of the absorption times. We point out the difficulty in establishing the important properties of the MAT directly from the absorption time expression, obtained by using a conventional method. This difficulty is then alleviated by transforming the absorption time into the spacing rule domain. The transformation offers insights into the properties of the absorption time and the MAT.

We first derive an explicit expression for the absorption time in the spacing rule domain. We then show that the absorption time is convex in $\mathbb{R}^{N_{\rm hit}}$, rotation invariant, and reversal invariant. We show that the minimum MAT over all possible search orders is lower bounded by the solution to an integer programming problem whose fluid approximation (or relaxation) has a convex objective function. Thus, well known techniques in convex optimization can be used to find the explicit solution algorithmically. We also derive the upper bound on the MAT over all possible search orders. The upper bound expression is explicit and depends on the details of the signal detection procedure. We further show that the MAT of the FSSS does not depend on the location of the first in-phase cell. Thus, the evaluation of the MAT for the FSSS can be simplified significantly.

Note that the proofs of the lower bound (Theorem 4) and the upper bound (Theorem 5) on the MATs do not require the fact that propagation paths arrive in a single cluster. Hence, these bounds are valid for an environment in which multiple clusters of propagation paths are observed at the receiver. In our approach, we deliberately represent the details, such as the fading statistic, the receiver's architecture, and the design choice of decision variables, by a few parameters in order to gain insights into important properties of acquisition receivers. The results in this paper can be applied to the design and analysis of fast acquisition systems in various wideband scenarios, including a broad class of fading conditions, hardware implementations, and operating environments.

Appendix I

Nonnegative Definiteness of the Hessian Matrix $\mathbf{H}$

Lemma 1 (Nonnegative Definite Kernel): Let $T > 0$ and $-e^{-2T} < \theta < e^{-2T}$ be given. Define a kernel

$$K(s,t) \triangleq \theta e^{|s-t|} + e^{-|s-t|} \tag{24}$$

for $-T \le s,t \le T$. Then, $K(s,t)$ is nonnegative definite on the space $[-T,T]\times[-T,T]$. That is, $\int_{-T}^{T}\int_{-T}^{T} f(s)K(s,t)f(t)\,ds\,dt \ge 0$ for any function $f(t) \in \mathcal{L}^2$.

Proof: For any $s,t \in [-T,T], K(s,t) = K(t,s)$ and $|K(s,t)| \le 2$. Thus, $K(s,t)$ is symmetric and square-integrable (i.e., $\int_{-T}^{T}\int_{-T}^{T} |K(s,t)|^2\,ds\,dt < \infty$). By [[37], Thm 7.71, p. 127], $K(s,t)$ is nonnegative definite on $[-T,T]\times[-T,T]$ iff all eigenvalues of $K(s,t)$ are positive. We will now derive a complete set of eigenvalues of $K(s,t)$ and show that they are positive.

The eigenvalues λ_i satisfy the following integral equation

$$\lambda_i \varphi_i(s) = \int_{-T}^{T} K(s,t)\varphi_i(t)\,dt, \quad -T \le s \le T \tag{25}$$

where $\varphi_i(t)$ are orthonormal eigenfunctions corresponding to the eigenvalues λ_i.[12] Substituting $K(s,t)$ and separating the integral into the sum of two integrals, we have

$$\lambda\varphi(s) = \int_{-T}^{s} (\theta e^{s-t} + e^{t-s})\varphi(t)\,dt + \int_{s}^{T} (\theta e^{t-s} + e^{s-t})\varphi(t)\,dt. \tag{26}$$

Using Leibniz's Rule to differentiate (26) with respect to s once, we have

$$\lambda\varphi'(s) = \int_{-T}^{s} (\theta e^{s-t} - e^{t-s})\varphi(t)\,dt + \int_{s}^{T} (-\theta e^{t-s} + e^{s-t})\varphi(t)\,dt. \tag{27}$$

Differentiating (26) twice, we have

$$\lambda\varphi''(s) = \left[\int_{-T}^{s} (\theta e^{s-t} + e^{t-s})\varphi(t)\,dt + \int_{s}^{T} (\theta e^{t-s} + e^{s-t})\varphi(t)\,dt\right] - 2(1-\theta)\varphi(s). \tag{28}$$

The sum of the above two integrals in the brackets is simply $\lambda\varphi(s)$. Therefore, the eigenfunction satisfies the second order differential equation

$$\lambda\varphi''(s) = [\lambda - 2(1-\theta)]\varphi(s). \tag{29}$$

We consider four separate cases.

[12]The existence of countably many eigenvalues of $K(s,t)$ follows from the theorem of Hilbert and Schmidt [[38], p. 243].

1) The eigenvalue is zero: $\lambda = 0$.
Equation (29) implies that $\varphi(s) = 0$, for $s \in [-T, T]$, which is not an eigenfunction. Therefore, this case is impossible.
2) The eigenvalue is nonzero and the coefficient satisfies

$$\left(\frac{\lambda - 2(1-\theta)}{\lambda}\right) > 0. \tag{30}$$

In this case, solutions to the differential equation (29) are of the form

$$\varphi(s) = Ae^{\nu s} + Be^{-\nu s} \tag{31}$$

where ν is defined to be $\nu \triangleq \sqrt{(\lambda - 2(1-\theta))/\lambda}$. Substituting (31) into the integral equation in (25), we have

$$\begin{aligned}\lambda Ae^{\nu s} &+ \lambda Be^{-\nu s}\\ &= \left[\frac{2(1-\theta)A}{1-\nu^2}\right]\cdot e^{\nu s} + \left[\frac{2(1-\theta)B}{1-\nu^2}\right]\cdot e^{-\nu s}\\ &+ \frac{1}{1-\nu^2}\cdot [A(1+\nu)(\theta e^{T-\nu T} - e^{-T+\nu T})\\ &\qquad + B(1-\nu)(\theta e^{T+\nu T} - e^{-T-\nu T})]\cdot e^{s}\\ &+ \frac{1}{1-\nu^2}\cdot [A(1-\nu)(\theta e^{T+\nu T} - e^{-T-\nu T})\\ &\qquad + B(1+\nu)(\theta e^{T-\nu T} - e^{-T+\nu T})]\cdot e^{-s}.\end{aligned} \tag{32}$$

Because the functions $\{e^s, e^{-s}, e^{\nu s}, e^{-\nu s}\}$ are linearly independent, we compare their coefficients on the left- and right-hand sides of (32) and conclude that

$$\lambda = \frac{2(1-\theta)}{1-\nu^2}, \tag{33a}$$

$$\begin{aligned}0 = {}& A(1+\nu)(\theta e^{T-\nu T} - e^{-T+\nu T})\\ &+ B(1-\nu)(\theta e^{T+\nu T} - e^{-T-\nu T})\end{aligned} \tag{33b}$$

$$\begin{aligned}0 = {}& B(1+\nu)(\theta e^{T-\nu T} - e^{-T+\nu T})\\ &+ A(1-\nu)(\theta e^{T+\nu T} - e^{-T-\nu T}).\end{aligned} \tag{33c}$$

The unknowns A and B can be solved for fixed T, θ, and ν using the linear equations (33b) and (33c).
Adding (33b) and (33c) yields, after some algebra,

$$\begin{aligned}0 = (A+B)\cdot [&\nu(\theta e^{T} + e^{-T})(e^{2\nu T} - 1)\\ &+ (e^{-T} - \theta e^{T})(e^{2\nu T} + 1)].\end{aligned} \tag{34}$$

Subtracting (33c) from (33b) yields, after some algebra,

$$\begin{aligned}0 = (A-B)\cdot [&(\theta e^{T} + e^{-T})(e^{2\nu T} - 1)\\ &+ \nu(e^{-T} - \theta e^{T})(e^{2\nu T} + 1)].\end{aligned} \tag{35}$$

Recall that T, θ, and ν satisfy the conditions $T > 0, -e^{-2T} < \theta < e^{-2T}$, and $\nu > 0$, implying that the quantities in the square brackets of (34) and (35) are strictly positive. Thus, (34) and (35) reduce to the system of equations

$$0 = A + B$$

$$0 = A - B$$

which has the solution $A = B = 0$. Substituting the values of A and B into (31) yields $\varphi(s) = 0$, for $s \in [-T, T]$, which is an invalid eigenfunction. Therefore, the case in (30) is impossible.
3) The eigenvalue is nonzero and the coefficient satisfies

$$\left(\frac{\lambda - 2(1-\theta)}{\lambda}\right) = 0. \tag{36}$$

Thus, solutions to (29) are of the form

$$\varphi(s) = As + B. \tag{37}$$

Substituting (31) into the integral equation in (25), we have

$$\begin{aligned}\lambda As &+ \lambda B\\ &= [2(1-\theta)A]\cdot s + [2(1-\theta)B]\\ &\quad + [\theta e^{T}(A + B - AT) + e^{-T}(-A - B - AT)]\cdot e^{s}\\ &\quad + [\theta e^{T}(-A + B + AT) + e^{-T}(A - B + AT)]\cdot e^{-s}.\end{aligned} \tag{38}$$

Because the functions $\{1, s, e^s, e^{-s}\}$ are linearly independent, we compare their coefficients on the left- and right-hand sides of (38) and conclude that

$$\lambda = 2(1-\theta) \tag{39a}$$

$$0 = \theta e^{T}(A + B - AT) + e^{-T}(-A - B - AT) \tag{39b}$$

$$0 = \theta e^{T}(-A + B + AT) + e^{-T}(A - B + AT). \tag{39c}$$

Equations (39b) and (39c) are linear in A and B and have two unknowns A and B. Adding (39b) and (39c) yields the condition

$$0 = B(e^{-T} - \theta e^{T}).$$

Since $-e^{-2T} < \theta < e^{-2T}$, the quantity in the parentheses is strictly positive, and thus $B = 0$. Equating (39b) and (39c) and substituting $B = 0$ yield the condition

$$0 = A(e^{-T}(1+T) - e^{T}\theta(1-T)).$$

Considering $(1-T) \leq 0$ and $(1-T) > 0$ separately, we can easily verify that the quantity in the parentheses is positive, and thus $A = 0$. As a result, (39b) and (39c) have a unique solution $A = B = 0$. Then the function $\varphi(\cdot)$ becomes $\varphi(s) = 0$, for $s \in [-T, T]$, which is an invalid eigenfunction. Therefore, the case in (36) is impossible.
4) The eigenvalue is nonzero and the coefficient satisfies

$$\left(\frac{\lambda - 2(1-\theta)}{\lambda}\right) < 0. \tag{40}$$

Thus, solutions to (29) are of the form

$$\varphi(s) = A\cos(\mu s) + B\sin(\mu s) \tag{41}$$

where μ is defined to be $\mu \triangleq \sqrt{-(\lambda - 2(1-\theta))/\lambda}$. Substituting (31) into the integral equation in (25), we have

$$\lambda A\cos(\mu s) + \lambda B\sin(\mu s)$$

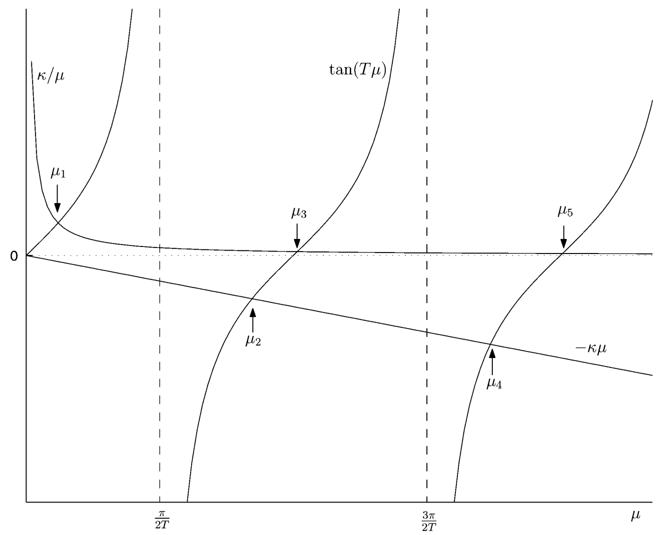

Fig. 6. The solutions $\mu_i > 0$ to the transcendental equation can be found graphically.

$$= \left[\frac{2(1-\theta)}{1+\mu^2}A\right]\cdot\cos(\mu s) + \left[\frac{2(1-\theta)}{1+\mu^2}B\right]\cdot\sin(\mu s)$$
$$+ [(\sin\mu T)(B-\mu A)(\theta e^T + e^{-T})$$
$$- (\cos\mu T)(B\mu + A)(\theta e^T - e^{-T})]\cdot e^s$$
$$+ [(\cos\mu T)(B\mu - A)(\theta e^T - e^{-T})$$
$$- (\sin\mu T)(B+\mu A)(\theta e^T + e^{-T})]\cdot e^{-s}. \tag{42}$$

Comparing the coefficients of $\cos(\mu s), \sin(\mu s), e^s$, and e^{-s} on the left- and right-hand sides, we have the constraints

$$\lambda = \frac{2(1-\theta)}{1+\mu^2} \tag{43a}$$

$$\tan\mu T = -\left(\frac{B\mu + A}{B - A\mu}\right)\kappa \tag{43b}$$

$$\tan\mu T = -\left(\frac{B\mu - A}{B + A\mu}\right)\kappa \tag{43c}$$

in which $\kappa \triangleq (e^{-T} - \theta e^T)/(e^{-T} + \theta e^T) > 0$. By equating the expressions for $\tan\mu T$ in (43), we conclude that $AB = 0$. Therefore, the eigenvalues are given by

$$\lambda_i = \frac{2(1-\theta)}{1+\mu_i^2}, \quad i = 1, 2, 3, \ldots \tag{44}$$

in which μ_i are solutions to the transcendental equation

$$(\tan\mu T + \kappa\mu)\left(\tan\mu T - \frac{\kappa}{\mu}\right) = 0.$$

We note that the solution $\mu_i > 0$ can be found graphically (see Fig. 6). Clearly, the eigenvalues in (44) are positive.

Because all eigenvalues of $K(s,t)$ are strictly positive, $K(s,t)$ is a nonnegative definite kernel. That completes the proof. □

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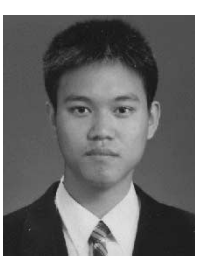

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