Multipath Aided Rapid Acquisition: Optimal Search Strategies

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***Abstract*—In this paper, we propose a search technique that takes advantage of multipath, which has long been considered deleterious for efficient communication, to aid the sequence acquisition in dense multipath channels. We consider a class of serial-search strategies and use optimization and convexity theories to determine fundamental limits of achievable mean acquisition times (MATs). In particular, we derive closed-form expressions for both the minimum and maximum MATs and the conditions for achieving these limits. We prove that a fixed-step serial search, a form of nonconsecutive serial search, achieves a near-optimal MAT. We also prove that the conventional serial search, in which consecutive cells are tested serially, should be avoided as it results in the maximum MAT. Our results are valid for all signal-to-noise ratio (SNR) values, regardless of the specifics of the detection layer and the fading distributions.**

***Index Terms*—Acquisition, dense multipath channels, nonconsecutive serial search, spread spectrum.**

I. Introduction

SEQUENCE synchronization is an important task for a spread spectrum receiver. Before communication commences, the receiver must search for a location of sequence phase within a required accuracy. The synchronization process occurs in two stages: the acquisition stage (the focus of this paper) and the tracking stage [1]–[4]. During the acquisition stage, the receiver coarsely aligns the sequence of the locally generated reference (LGR) with that of the received signal. The receiver then enters the tracking stage to finely align the two sequences and maintain the synchronization throughout the communication. It has been shown that an acquisition problem is a hypothesis testing problem [5].

The total number $N_{\rm unc}$ of phases or cells that the receiver needs to test depends on the temporal uncertainty range $[T_{\rm begin}, T_{\rm end})$ of a phase delay and the resolution $T_{\rm res}$ to resolve the phase delay.[1] The expression for $N_{\rm unc}$ is given by

$$N_{\rm unc} = \frac{T_{\rm end} - T_{\rm begin}}{T_{\rm res}} \tag{1}$$

Manuscript received December 10, 2004; revised May 2, 2006. This work was supported, in part, by the Office of Naval Research Young Investigator Award N00014-03-1-0489, the National Science Foundation under Grants ANI-0335256 and ECS-0636519, DoCoMo USA Labs, and the Charles Stark Draper Laboratory Robust Distributed Sensor Networks Program. The material in this paper was presented, in part, at the Conference on Information Sciences and Systems, Baltimore, MD, March 2005, at the IEEE International Conference on Utra-Wideband, Zürich, Switzerland, September 2005, and at the Information Theory and Application—Inaugural Workshop, La Jolla, CA, February 2006.

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Communicated by A. Høst-Madsen, Associate Editor for Detection and Estimation.

Color versions of Figs. 1–12 are avaiable online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TIT.2006.887509

[1]Subscript "unc" stands for uncertainty.

which can range from a few cells to several thousand cells, depending on the application [6]. Without loss of generality, we index the cells from 1 to $N_{\rm unc}$. Cell i, $1 \leq i \leq N_{\rm unc}$, corresponds to a hypothesized phase delay in the range $[T_{\rm begin} + (i-1)T_{\rm res}, T_{\rm begin} + iT_{\rm res})$. An uncertainty index set

$$\mathcal{U} = \{1, 2, 3, \ldots, N_{\rm unc}\} \tag{2}$$

denotes a collection of cells to test. Because the ratio $1/T_{\rm res}$ is proportional to the transmission bandwidth, $N_{\rm unc}$ can be very large, especially, for a wide-bandwidth transmission system [7]. In that scenario, acquisition of a received signal in a reasonable amount of time is a challenging task.

Unlike an additive white Gaussian noise (AWGN) channel, a dense urban or indoor channel provides us with multiple propagation paths, which can be resolved via the use of wide-bandwidth signals [8], [9]. The number of correct phases or in-phase cells, denoted by $N_{\rm hit}$, in a dense multipath channel is proportional to the number of resolvable paths. Multiple resolvable paths tend to arrive in a cluster in dense multipath channels [10]–[15], giving rise to consecutive in-phase cells, modulo-$N_{\rm unc}$, in the uncertainty index set.

Designing an acquisition system involves two broad design aspects. One deals with how the decision is made at the detection layer. Examples of the relevant issues at this layer include combining methods for decision variables, the number of stages in a multidwell detector, a design choice for decision thresholds, and the evaluation of the detection and false-alarm probabilities. The other aspect deals with the procedure for finding a correct cell at the search layer. Examples of the relevant issues include:

- the choice of search strategy (e.g., serial search [6], fully parallel search [16], or hybrid search [17]), and
- the selection of efficient search order (the sequence in which cells are tested).

In general, the goal of the acquisition receiver is to find a correct sequence phase as fast as possible.

The performance of the acquisition system is measured typically by the mean acquisition time (MAT), the average duration required for the receiver to achieve acquisition. A common method for finding the MAT is to use a flow diagram. A flow diagram that describes the acquisition procedure in AWGN channels [18]–[20] or in frequency-nonselective fading channels [21]–[27] simply has one in-phase cell. On the other hand, in multipath fading channels, the flow diagram has multiple in-phase cells corresponding to the multiple resolvable paths [28]–[35].

There are two major approaches to improve the MAT. The first approach improves the MAT at the detection layer. For example, a receiver may dedicate more resources, such as correlators, to form a decision variable [33]–[38], use passive correla-

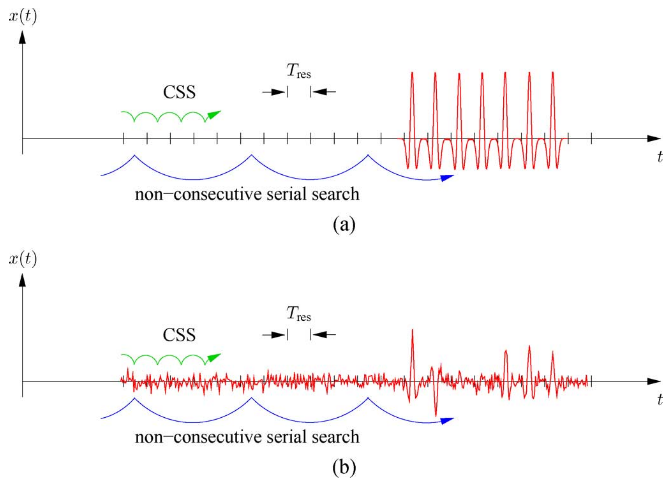

Fig. 1. A correlator output $x(t)$ contains several resolvable peaks. (a) An idealized scenario in the absence of fading and noise. (b) A realistic scenario in the presence of fading and noise.

tors to increase a decision rate [19], use an appropriate decision rule [39], [40], or employ sequential techniques [41]–[45]. The second approach improves the MAT at the search layer. For example, a receiver may perform a hybrid search by using multiple correlators [46]–[49] or use a special search pattern such as an expanding zig-zag window [2], [20], [50], [51] or a nonconsecutive [29]–[33] serial search.

In a few special cases, the MAT of the conventional serial search (CSS)[2] is shown to be longer than that of a nonconsecutive serial search (see also [29]–[33]). To gain some insight into this behavior, let us consider an idealized scenario in the absence of fading and noise. In this hypothetical scenario (see Fig. 1(a)), a receiver that skips some cells after each test will reach and find an in-phase cell faster than does the receiver that uses the CSS. This example indicates that multipath helps the signal acquisition.

While the idealized scenario in the previous example gives credence to the idea that multipath can be useful, the intuition gained from the example becomes questionable in a realistic scenario. In the presence of fading and noise (see Fig. 1(b)), the receiver makes erroneous decisions when testing an in-phase or a non-in-phase cell. It is unclear whether the nonconsecutive serial search outperforms the conventional serial search in every operating environment.

This paper will focus on search techniques that exploit multipath to aid acquisition in dense urban or indoor channels. We consider acquisition receivers that can test cells in arbitrary orders using active correlators.[3] Our goal is to investigate the following questions.

- What are the fundamental limits of the achievable MATs? In other words, what are the minimum and maximum MATs over all possible search orders?
- What are the search orders that achieve the minimum MAT?
- What are the search orders that result in the maximum MAT?

We focus on the most commonly used search strategy, namely, the serial search [18]–[21], [32]–[35], and our analysis employs a *nonpreferential* flow diagram.[4] The key contributions of this paper are as follows.

- We introduce the concept of a *spacing rule*, which describes the structure of the flow diagram, and derive the absorption time[5] expression as an implicit function of a *spacing rule*. We then derive the optimal *spacing rule* by using convexity and optimization theories.
- We derive bounds for the minimum MAT and find a search order that yields a near-optimal MAT.
- We derive the explicit expression for the maximum MAT and show that the CSS yields this maximum.

Our results are valid for all values of signal-to-noise ratio (SNR), regardless of the decision rules of the detection layer or the operating environments.

This paper is organized as follows. In Section II, we present the system model and basic definitions for the acquisition system. In Section III, we derive the absorption time as a function of the spacing rule and prove important properties of the absorption time. In Section IV, we derive the explicit expression of a lower bound for the MAT and find the search order that yields the near-optimal MAT. In Section V, we derive the explicit expression for the maximum MAT and prove that the CSS results in the maximum MAT. Finally, the important findings are summarized in Section VI.

II. System Model and Basic Definitions

The set of all possible search orders is denoted by

$$\mathcal{P} = \left\{\pi \,\middle|\, \pi\colon \mathcal{U} \to \mathcal{U} \text{ is a permutation function and } \pi(1) = 1\right\}. \tag{3}$$

An element π of $\mathcal{P}$ is called a search order, which will be sometimes written as an N_{unc}-tuple

$$[\pi(1), \pi(2), \ldots, \pi(N_{\text{unc}})].$$

This N_{unc}-tuple emphasizes the order

$$\pi(k), \pi(k+1), \ldots, \pi(N_{\text{unc}}), \pi(1), \pi(2), \ldots, \pi(N_{\text{unc}}), \pi(1), \pi(2), \ldots.$$

in which the receiver tests the cells, where $\pi(k) \in \mathcal{U}$ is the first cell to be interrogated. We note that $\pi(k)$ can be any cell and the search order itself does not specify which cell to test first. Some common search orders that have been used in the literature are shown in Fig. 2.

The CSS [34], [35], where the consecutive cells are tested serially, yields the N_{unc}-tuple $[1, 2, 3, \ldots, N_{\text{unc}}]$ with the corresponding search order

$$\pi^1(k) = k. \tag{4}$$

[2]The CSS is a search order that tests consecutive cells serially.

[3]Note that the inability of passive correlators to test cells in arbitrary orders excludes them from this study.

[4]The definition of a nonpreferential flow diagram is given in Section II

[5]The absorption time is the average time to transit from a start state to an absorbing state in a Markov flow diagram.

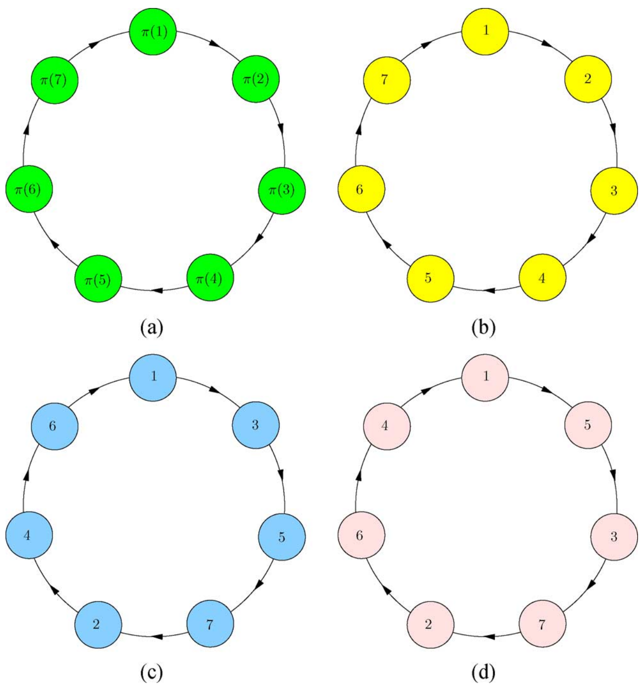

Fig. 2. A receiver tests the cells according to the search order. (a) A generic search order π. (b) The search order π^1 of the CSS. (c) The search order π^2 of the fixed-step serial search (FSSS) with the step size $N_{\rm J} = 2$. (d) The search order $\pi_{\rm R}$ of the bit-reversal serial search.

The fixed-step serial search (FSSS) [32], [33], [52], where the receiver skips $N_{\rm J} \geq 1$ cells before it performs the next test, yields the $N_{\rm unc}$-tuple[6]

$$[1, 1 \oplus N_{\rm J}, 1 \oplus 2N_{\rm J}, \ldots, 1 \oplus (N_{\rm unc} - 1)N_{\rm J}]$$

with the corresponding search order

$$\pi^{N_{\rm J}}(k) = 1 \oplus (k-1)N_{\rm J}. \tag{5}$$

To ensure that the mapping $\pi^{N_{\rm J}}:\mathcal{U} \rightarrow \mathcal{U}$ is a bijection, we require that $N_{\rm J}$ and $N_{\rm unc}$ be relatively prime. Clearly, the CSS π^1 is a special case of the FSSS $\pi^{N_{\rm J}}$ with the step size $N_{\rm J} = 1$.

The bit-reversal serial search is proposed in [32] and corresponds to

$$[\pi_{\rm R}(1) = 1, \pi_{\rm R}(2), \ldots, \pi_{\rm R}(N_{\rm unc})]$$

where elements $\pi_{\rm R}(\cdot)$ are defined relatively to one another as follows. For $i \neq j$

$$\pi_{\rm R}(i) < \pi_{\rm R}(j) \Leftrightarrow \text{rev}(i) < \text{rev}(j) \tag{6}$$

where $\text{rev}(i)$ is the reversal of the $\lceil \log_2 N_{\rm unc} \rceil$ binary digit representation of the integer $i-1$. Equation (6) specifies the unique order of $N_{\rm unc}$ cells in the uncertainty index set: assigning the cost $\text{rev}(i)$ to cell i and arranging the cells in ascending order according to their costs.

In general, there are $(N_{\rm unc} - 1)!$ different search orders, and it is imperative to find the one that minimizes the MAT. For a given search order, the MAT can be evaluated by using a flow diagram.

Fig. 3 depicts a flow diagram, which corresponds to a serial-search strategy with a generic search order π. There are $N_{\rm unc}+1$ states totally: one absorbing state (ACQ), $N_{\rm hit}$ states of type H_1, and $N_{\rm unc} - N_{\rm hit}$ states of type H_0. The ACQ state represents the event of successful acquisition. Each of the $N_{\rm hit}$ states of type H_1 corresponds to an in-phase cell, while each of the remaining $N_{\rm unc} - N_{\rm hit}$ states of type H_0 corresponds to a non-in-phase cell. The disjoint union of the in-phase and non-in-phase cells forms an uncertainty index set $\mathcal{U} = \{1, 2, 3, \ldots, N_{\rm unc}\}$.

The location B of the first in-phase cell is unknown to the receiver.[7] We treat B as a random variable, uniformly distributed on $\mathcal{U}$. In dense multipath environments, such as ultra-wide bandwidth (UWB) indoor or urban channels, multipath tends to arrive in a cluster [10]–[15]. In this case, conditioned on $\mathsf{B} = b_0$, the index set corresponding to the in-phase cells $\mathcal{H}_{\rm hit}(b_0) \subset \mathcal{U}$ is[8]

$$\mathcal{H}_{\rm hit}(b_0) \triangleq \{b_0, b_0 \oplus 1, \ldots, b_0 \oplus (N_{\rm hit} - 1)\}. \tag{7}$$

Since there is no *a priori* knowledge of B, the receiver may begin the search at any cell K. Therefore, we also consider K to be a uniform random variable over $\mathcal{U}$.

The bijective property of π implies that there are exactly $N_{\rm hit}$ unique integers $1 \leq k_1 < k_2 < \cdots < k_{N_{\rm hit}} \leq N_{\rm unc}$ such that

$$\{\pi(k_1), \pi(k_2), \ldots, \pi(k_{N_{\rm hit}})\} = \mathcal{H}_{\rm hit}(b_0).$$

In a flow diagram, those in-phase cells $\pi(k_i)$ have paths to the absorbing state.

Let $H_{\rm D}(z)$, $H_{\rm M}(z)$, and $H_0(z)$, respectively, denote generic path gains from an H_1-state to ACQ, from an H_1-state to the adjacent nonabsorbing state, and from an H_0-state to the adjacent nonabsorbing state. These path gains can be determined from the details in the detection layer [18]–[20]. It was shown in [18, eq (7b)] that the absorption time depends on the detection layer through $H_i(1)$ and $H_i'(1)$, for $i \in \{{\rm D}, {\rm M}, 0\}$. This result implies that, for the purpose of MAT calculation, a path gain $H_i(z)$ can be replaced by an equivalent path gain, say $F_i(z)$, as long as $H_i(1) = F_i(1)$ and $H_i'(1) = F_i'(1)$. Therefore, the path gains $H_{\rm D}(z)$, $H_{\rm M}(z)$, and $H_0(z)$ can be represented equivalently by $P_{\rm D} z^{\tau_{\rm D}}$, $P_{\rm M} z^{\tau_{\rm M}}$, and $z^{\tau_{\rm P}}$, respectively, where

$$\begin{aligned} P_{\rm D} &= H_{\rm D}(1) \\ \tau_{\rm D} &= \begin{cases} H_{\rm D}'(1)/H_{\rm D}(1), & H_{\rm D}(1) \neq 0 \\ 1, & H_{\rm D}(1) = 0 \end{cases} \\ P_{\rm M} &= H_{\rm M}(1) \\ \tau_{\rm M} &= \begin{cases} H_{\rm M}'(1)/H_{\rm M}(1), & H_{\rm D}(1) \neq 0 \\ 1, & H_{\rm M}(1) = 0 \end{cases} \\ \tau_{\rm P} &= H_0'(1). \end{aligned} \tag{8}$$

The parameters $P_{\rm D} z^{\tau_{\rm D}}$, $P_{\rm M} z^{\tau_{\rm M}}$, and $z^{\tau_{\rm P}}$ can be interpreted as effective detection layer parameters. Specifically, the receiver

[6] The symbol $\oplus$ denotes the modulo-$N_{\rm unc}$ addition defined by $x \oplus y \triangleq x + y - lN_{\rm unc}$, for some unique integer l such that $x + y - lN_{\rm unc} \in \mathcal{U}$. We will write $x \ominus y$ for $x \oplus (-y)$.

[7] A sanserif font denotes a random variable.

[8] In some scenarios, multiple clusters of propagation paths are observed at the receiver. We will discuss such cases in the conclusion section.

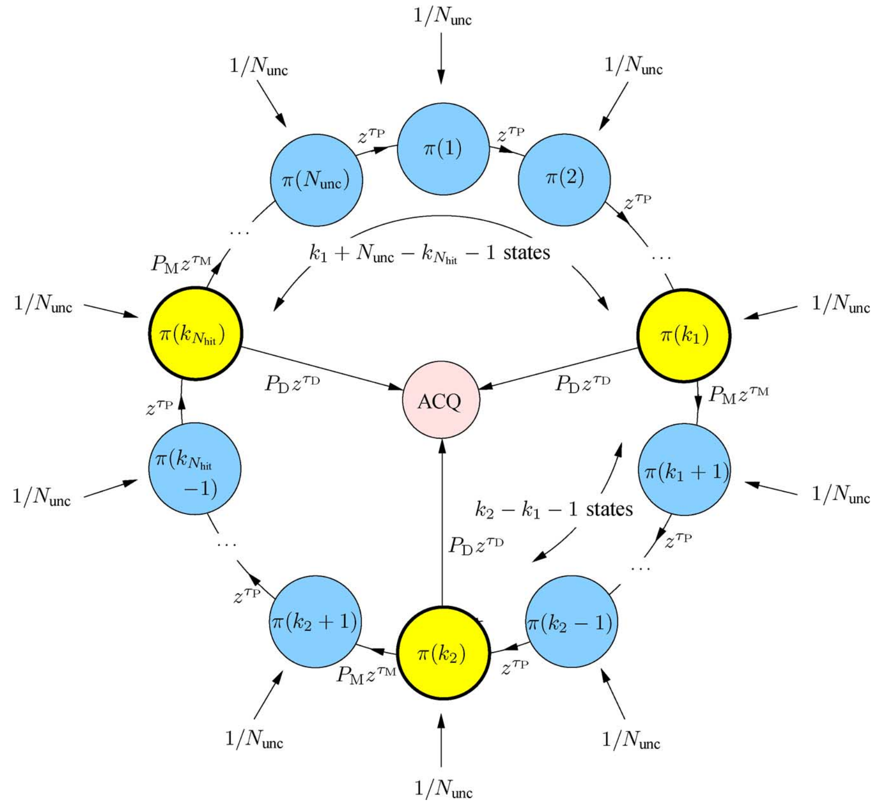

Fig. 3. A flow diagram for the serial search with the search order π contains $N_{\rm unc} + 1$ states. The state labeled ACQ is the absorbing state. The states in thick circles are H_1-states. The remaining states are H_0-states.

spends $\tau_{\rm D}$ time units and makes the correct decision with probability $P_{\rm D}$ when testing an in-phase cell. The receiver spends $\tau_{\rm M}$ time units and makes the incorrect decision with probability $P_{\rm M} = 1 - P_{\rm D}$ when testing an in-phase cell. The receiver spends $\tau_{\rm P}$ time units to eventually make the correct decision when testing a non-in-phase cell.

The structure of the flow diagram describes the arrangement of the in-phase and non-in-phase cells. This structure plays an important role in the acquisition system as it strongly influences the absorption time and the MAT. The structure of a flow diagram can be described by its *description*.

Definition 1 (Description): A *description* is a tuple (π, b) of the search order π and the location b of the first in-phase cell. The set of all possible descriptions is denoted by

$$\mathcal{D} = \mathcal{P} \times \mathcal{U}. \tag{9}$$

The description (π, b) characterizes the structure of a flow diagram. In particular, π specifies the order of the nonabsorbing states, while b determines the set $\mathcal{H}_{\rm hit}(b)$ of states that have transition edges to the absorbing state. We now focus our attention on a widely used class of flow diagrams [18]–[20], [32], [33], which we refer to as a *nonpreferential* flow diagram (see Fig. 3).

Definition 2 (Nonpreferential Flow Diagram): The flow diagram is *nonpreferential* if it has the following properties:

1) the probability of starting the search at any nonabsorbing state is equally likely;
2) every path to the absorbing state has the same path gain;
3) every path from an H_0-state has the same path gain; and
4) every path from an H_1-state to the adjacent nonabsorbing state has the same path gain.

The use of a nonpreferential flow diagram is reasonable when the power dispersion profile (PDP) is decaying slowly or constant over an interval. Indeed, constant PDPs have been used to study various aspects of spread spectrum systems [8], [9], [53]–[56]. Propagation measurements in urban and suburban environments [57]–[59] and mountainous terrain [60] exhibit characteristics supporting such a PDP since they show channels with energy spread over a continuum of arrival times. Thus, a nonpreferential flow diagram serves as a basic model for analyzing the performance of an acquisition system operating in dense multipath environments.

III. The Absorption Time

A. Conventional Approach

For a given search order π, the MAT is given by

$$\begin{aligned}\mathbb{E}\{\mathsf{T}_{\rm ACQ}(\pi)\} &= \sum_{b=1}^{N_{\rm unc}} \sum_{k=1}^{N_{\rm unc}} \mathbb{E}\{\mathsf{T}_{\rm ACQ}(\pi) \,|\, \mathsf{B} = b, \mathsf{K} = k\} \\ &\quad \times \Pr\{\mathsf{B} = b\} \Pr\{\mathsf{K} = k\} \qquad (10)\\ &= \frac{1}{N_{\rm unc}} \sum_{b=1}^{N_{\rm unc}} f(\pi, b).\end{aligned}$$

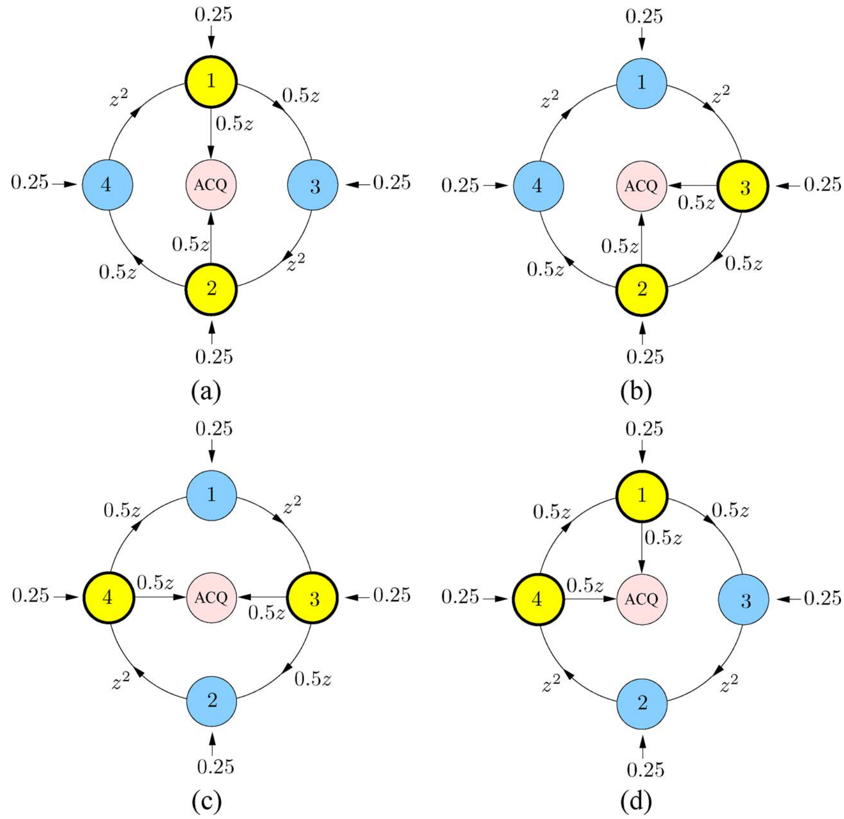

Fig. 4. Each flow diagrams corresponds to the bit-reversal search and has the following parameters: $N_{\rm unc} = 4$, $N_{\rm hit} = 2$, $P_{\rm D} = 0.5$, $\tau_{\rm D} = \tau_{\rm M} = 1$, and $\tau_{\rm P} = 2$. The structure of the flow diagram varies with the location B of the first in-phase cell. (a) $\mathsf{B} = 1$. (b) $\mathsf{B} = 2$. (c) $\mathsf{B} = 3$. (d) $\mathsf{B} = 4$.

Here, $f(\pi, b)$ is the absorption time of the flow diagram corresponding to the description (π, b)

$$f(\pi, b) \triangleq \sum_{k=1}^{N_{\rm unc}} \mathbb{E}\left\{\mathsf{T}_{\rm ACQ}(\pi) \,|\, \mathsf{B} = b, \mathsf{K} = k\right\} \Pr\left\{\mathsf{K} = k\right\}$$
$$\overset{\rm (a)}{=} \frac{1}{N_{\rm unc}} \frac{d}{dz} \left\{ \Big[\sum_{k=1}^{N_{\rm unc}} \sum_{i=0}^{N_{\rm unc}-1} H^b_{\pi(i \oplus k)}(z) \prod_{j=0}^{i-1} H^b_{\pi(j \oplus k)}(z) \Big] \Big/ \Big[1 - \prod_{i=1}^{N_{\rm unc}} G^b_i(z) \Big] \right\} \Bigg|_{z=1} \quad (11)$$

where

$$H^b_i(z) = \begin{cases} P_{\rm D} z^{\tau_{\rm D}}, & i \in \mathcal{H}_{\rm hit}(b) \\ 0, & \text{otherwise} \end{cases}$$

and

$$G^b_i(z) = \begin{cases} P_{\rm M} z^{\tau_{\rm M}}, & i \in \mathcal{H}_{\rm hit}(b) \\ z^{\tau_{\rm P}}, & \text{otherwise.} \end{cases}$$

The equality (a) follows from a loop-reduction technique, which is used to find the MATs in [18]–[20], [32], [33].

Remark 1: In general, averaging over B in (10) is required. Consider, for example, the flow diagrams in Fig. 4, corresponding to the bit reversal search $\pi_{\rm R}$ with different values of B. Equation (11) implies that the absorption time for each flow diagram is

$$f(\pi_{\rm R}, b) = \begin{cases} 5, & b = 1, 3 \\ 5\frac{1}{6}, & b = 2, 4. \end{cases}$$

Averaging over B gives the MAT of $5\frac{1}{12}$, which is distinct from any $f(\pi_{\rm R}, b)$. The example shows that in general the absorption time is a function of a particular value of B, and that the MAT calculation of [32], [33] under the assumption of $\mathsf{B} = 1$ may only give an approximation. In some special cases, averaging over B is unnecessary. For example, when the FSSS is employed, the absorption time does not depend on B, regardless of the step size (see [61] for the proof).

From (10), the MAT satisfies

$$\min_{(\pi,b) \in \mathcal{D}} f(\pi, b) \le \mathbb{E}\left\{\mathsf{T}_{\rm ACQ}(\pi_0)\right\} \le \max_{(\pi,b) \in \mathcal{D}} f(\pi, b) \quad (12)$$

for any given search order π_0. The above inequalities seem to give useful bounds on the MAT. However, the expression $f(\pi, b)$ in (11) does not reveal its dependence on the search order π explicitly. As a result, it is unclear how one can solve efficiently—if at all—the optimization problems $\min_{(\pi,b)\in\mathcal{D}} f(\pi, b)$ and $\max_{(\pi,b)\in\mathcal{D}} f(\pi, b)$.

To accentuate the need for more innovative and clever solutions to the optimization problem, we note that the direct approach that exhaustively searches over $\mathcal{P}$ for the best and worst search orders is impractical. Evaluation of the right-hand side of (10) for a given search order requires at least $N_{\rm unc}$ arithmetic operations to calculate $N_{\rm unc}$ absorption times. While the evaluation of (10) for a *given* π is feasible, the exhaustive search over *all* π on $\mathcal{P}$ requires at least $N_{\rm unc} \cdot |\mathcal{P}| = N_{\rm unc}!$ arithmetic

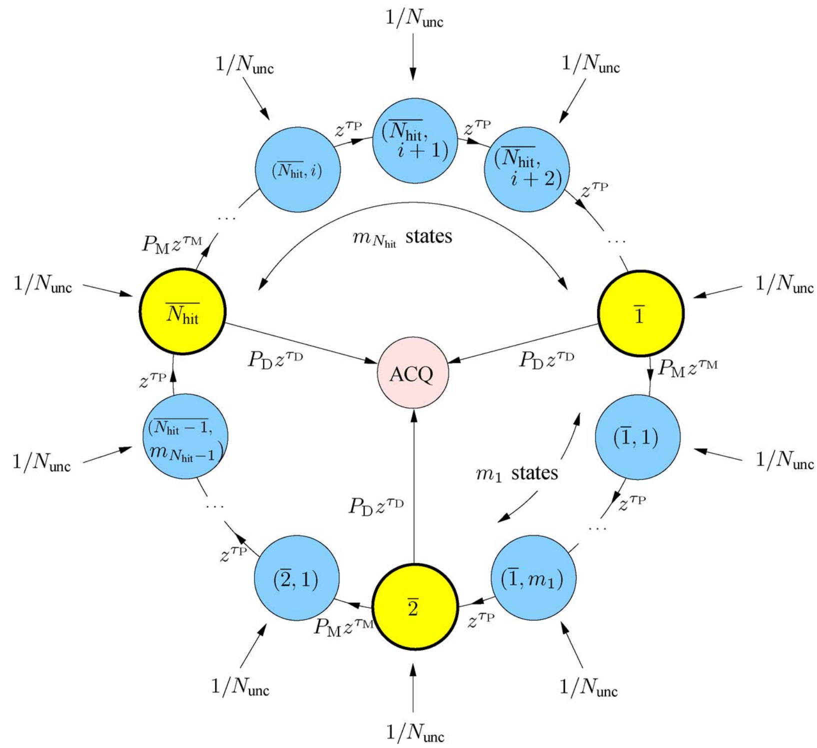

Fig. 5. The spacing rule $\boldsymbol{m} = (m_1, m_2, \ldots, m_{N_{\rm hit}})$ characterizes the structure of the flow diagram.

operations. For a small cardinality $N_{\rm unc} = 100$ of the uncertainty index set and a fictional machine that has a clock speed of 10^{20} Hz and performs one arithmetic operation per cycle, the exhaustive search requires more than 10^{130} years to complete. Clearly, the direct approach is extremely inefficient.

B. Transforming Into the Spacing Rule Domain

The difficulty associated with direct optimization can be alleviated by transforming the *descriptions* into the domain of *spacing rules*.[9]

Definition 3 (Spacing Rule): A *spacing rule* $\boldsymbol{m} = (m_1, m_2, \ldots, m_{N_{\rm hit}})$ of a nonpreferential flow diagram with $N_{\rm hit}$ H_1-states and $(N_{\rm unc} - N_{\rm hit})$ H_0-states is an element of the set[10]

$$\mathcal{S}_{\rm d} = \left\{(m_1, m_2, \ldots, m_{N_{\rm hit}}) \middle| \sum_{i=1}^{N_{\rm hit}} m_i = N_{\rm unc} - N_{\rm hit}; \forall i, m_i \in \mathbb{N}\right\}. \quad (13)$$

The spacing rule characterizes the structure of a nonpreferential flow diagram. In particular, the flow diagram has an H_1-state, which is followed by m_1 H_0-states, which are followed by another H_1-state, which is followed by m_2 H_0-states, and so on. The sum $\sum_{i=1}^{N_{\rm hit}} m_i$ must equal the number $N_{\rm unc} - N_{\rm hit}$ of H_0-states. Fig. 5 is the flow diagram with the spacing rule $\boldsymbol{m} = (m_1, m_2, \ldots, m_{N_{\rm hit}})$.

[9]Our approach follows the general philosophy of solving difficult problems in the transform domain [62], [63].

[10]The symbol $\mathbb{N}$ denotes the set of nonnegative integers, $\{0, 1, 2, 3, \ldots\}$.

Given the description (π, b), one can find the spacing rule via the mapping $\boldsymbol{s}\colon \mathcal{D} \to \mathcal{S}_{\rm d}$, such that

$$\boldsymbol{s}(\pi, b) \triangleq (m_1, m_2, \ldots, m_{N_{\rm hit}}) \quad (14)$$

where $m_i \triangleq k_{i+1} - k_i - 1$, for the unique integers

$$k_1 < k_2 < \cdots < k_{N_{\rm hit}} < k_{N_{\rm hit}+1} \triangleq k_1 + N_{\rm unc}$$

that satisfy $\{\pi(k_1), \pi(k_2), \ldots, \pi(k_{N_{\rm hit}})\} = \mathcal{H}_{\rm hit}(b)$. See Fig. 3 for illustration.

Fig. 6 shows the flow diagrams of the CSS π^1 when the first in-phase cells are $B = 1$ and $\mathsf{B} = N_{\rm unc} - N_{\rm hit} + 1$. The spacing rule corresponding to the description $(\pi^1, 1)$ is $(0, 0, \ldots, 0, N_{\rm unc} - N_{\rm hit})$, while the spacing rule corresponding to the description $(\pi^1, N_{\rm unc} - N_{\rm hit} + 1)$ is $(N_{\rm unc} - N_{\rm hit}, 0, 0, \ldots, 0)$. The set of spacing rules associated with the CSS is given by

$$\begin{aligned}\mathcal{E} &\triangleq \{\boldsymbol{s}(\pi^1, 1), \boldsymbol{s}(\pi^1, 2), \boldsymbol{s}(\pi^1, 3), \ldots, \boldsymbol{s}(\pi^1, N_{\rm unc})\} \\ &= \{\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}, \ldots, \boldsymbol{m}^{(N_{\rm hit})}\},\end{aligned} \quad (15)$$

where

$$\boldsymbol{m}^{(i)} \triangleq (0, 0, \ldots, 0, N_{\rm unc} - N_{\rm hit}, 0, 0, \ldots, 0) \quad (16)$$

denotes an $N_{\rm hit}$-dimensional vector with only one nonzero element at the ith component, $1 \leq i \leq N_{\rm hit}$. Note that $\mathcal{E} \subset \mathcal{S}_{\rm d}$, and each element of $\mathcal{E}$ describes the flow diagram with consecutive H_1-states and consecutive H_0-states.

Let $v(\boldsymbol{m})$ be the absorption time for the flow diagram corresponding to a spacing rule $\boldsymbol{m}$. Since both description $(\pi, b) \in \mathcal{D}$

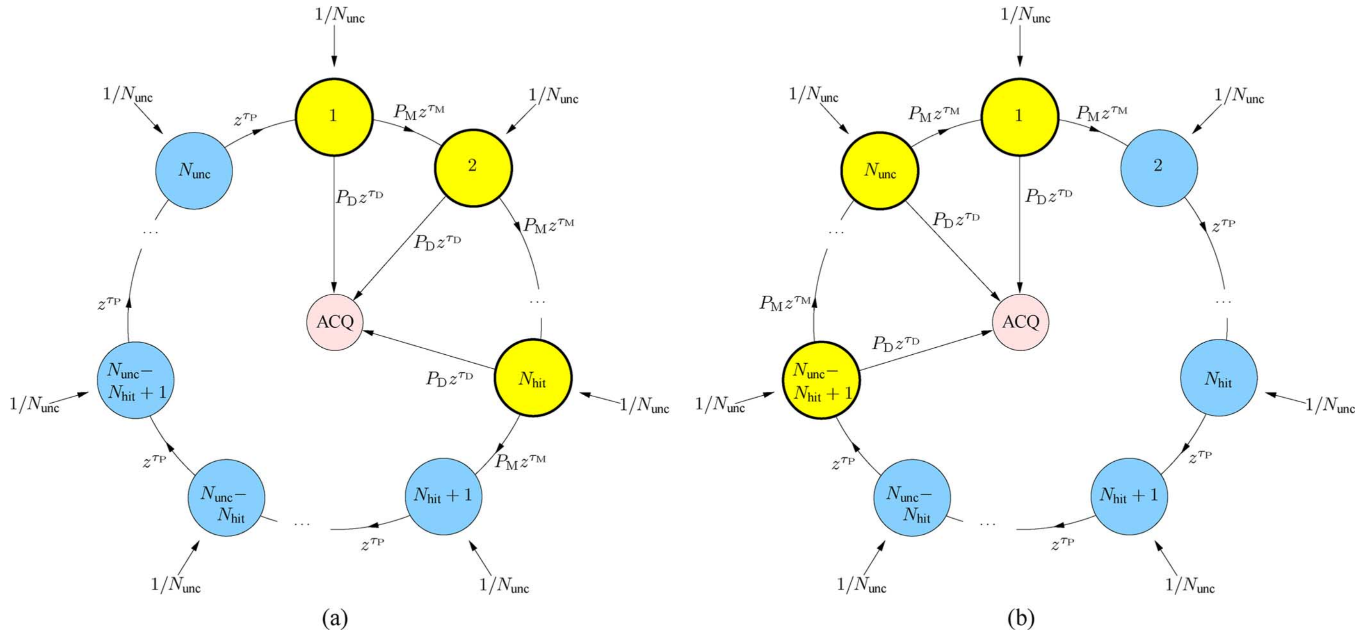

Fig. 6. Flow diagrams for the conventional serial search correspond to the different locations B of the first in-phase cell. (a) $\mathsf{B}=1$. (b) $\mathsf{B}=N_{\text{unc}}-N_{\text{hit}}+1$.

and spacing rule $\boldsymbol{m}\in\mathcal{S}_{\text{d}}$ characterize the structure of the flow diagram, which determines the absorption time, we have

$$\begin{aligned} f(\pi,b) &= v(\boldsymbol{s}(\pi,b)) \\ \min_{(\pi,b)\in\mathcal{D}} f(\pi,b) &= \min_{\boldsymbol{m}\in\mathcal{S}_{\text{d}}} v(\boldsymbol{m}) \\ \max_{(\pi,b)\in\mathcal{D}} f(\pi,b) &= \max_{\boldsymbol{m}\in\mathcal{S}_{\text{d}}} v(\boldsymbol{m}). \end{aligned} \tag{17}$$

Therefore, (12) is equivalent to

$$\min_{\boldsymbol{m}\in\mathcal{S}_{\text{d}}} v(\boldsymbol{m}) \le \mathbb{E}\left\{\mathsf{T}_{\text{ACQ}}(\pi_0)\right\} \le \max_{\boldsymbol{m}\in\mathcal{S}_{\text{d}}} v(\boldsymbol{m}). \tag{18}$$

Note that $\min_{\boldsymbol{m}\in\mathcal{S}_{\text{d}}} v(\boldsymbol{m})$ and $\max_{\boldsymbol{m}\in\mathcal{S}_{\text{d}}} v(\boldsymbol{m})$ are integer programming problems [64], [65].

C. Closed-Form Expression of $v(\boldsymbol{m})$

The goal of this subsection is to derive the explicit absorption time expression $v(\boldsymbol{m})$ for $\boldsymbol{m}\in\mathcal{S}_{\text{d}}$. Finding the absorption time reduces to simply solving a system of linear equations because the flow diagram has one absorbing state. The closed-form expression of $v(\boldsymbol{m})$ is given explicitly by the following theorem.

Theorem 1 (Absorption Time): The absorption time of the flow diagram with the spacing rule $\boldsymbol{m}\in\mathcal{S}_{\text{d}}$ is given by

$$v(\boldsymbol{m}) = A\sum_{i=1}^{N_{\text{hit}}} m_i^2 + \sum_{i=1}^{N_{\text{hit}}}\sum_{j=i+1}^{N_{\text{hit}}} B_{ij} m_i m_j + C \tag{19a}$$

$$= \frac{1}{2}\boldsymbol{m}^T\boldsymbol{H}\boldsymbol{m} + C \tag{19b}$$

where

$$A = \frac{\tau_{\text{P}}\left(1+{P_{\text{M}}}^{N_{\text{hit}}}\right)}{2N_{\text{unc}}\left(1-{P_{\text{M}}}^{N_{\text{hit}}}\right)} \tag{20}$$

$$B_{ij} = \frac{\tau_{\text{P}}\left({P_{\text{M}}}^{N_{\text{hit}}-(j-i)}+{P_{\text{M}}}^{j-i}\right)}{N_{\text{unc}}\left(1-{P_{\text{M}}}^{N_{\text{hit}}}\right)} \tag{21}$$

$$C = \left(1-\frac{N_{\text{hit}}}{N_{\text{unc}}}\right)\cdot\left(\frac{1+P_{\text{M}}}{1-P_{\text{M}}}\right)\frac{\tau_{\text{P}}}{2} + \frac{P_{\text{M}}}{1-P_{\text{M}}}\tau_{\text{M}} + \tau_{\text{D}} \tag{22}$$

$$\boldsymbol{H} = \frac{\tau_{\text{P}}}{N_{\text{unc}}\left(1-{P_{\text{M}}}^{N_{\text{hit}}}\right)}\left[{P_{\text{M}}}^{N_{\text{hit}}-|i-j|}+{P_{\text{M}}}^{|i-j|}\right]_{ij} \tag{23}$$

with $0^0 \triangleq 1$ and $\sum_{j=1}^{0} \triangleq 0$.

Proof: Let T_i denote the conditional absorption time, conditioned on the start location of the search at the H_1-state $\bar{i}$, $1\le i\le N_{\text{hit}}$. The states are labeled according to the convention in Fig. 5. Define $\alpha \triangleq P_{\text{M}}\tau_{\text{P}}$ and $\beta \triangleq P_{\text{D}}\tau_{\text{D}} + P_{\text{M}}\tau_{\text{M}}$. We have the relationship

$$\begin{aligned} T_1 &= P_{\text{D}}\tau_{\text{D}} + P_{\text{M}}(\tau_{\text{M}} + m_1\tau_{\text{P}} + T_2) \\ &= \beta + \alpha m_1 + P_{\text{M}}T_2 \\ T_2 &= \beta + \alpha m_2 + P_{\text{M}}T_3 \\ T_3 &= \beta + \alpha m_3 + P_{\text{M}}T_4 \\ &\vdots \\ T_{N_{\text{hit}}} &= \beta + \alpha m_{N_{\text{hit}}} + P_{\text{M}}T_1. \end{aligned}$$

Solving the above system of equations yields

$$T_i = \frac{\alpha}{1-{P_{\text{M}}}^{N_{\text{hit}}}}\cdot\left(\sum_{j=1}^{i-1}{P_{\text{M}}}^{N_{\text{hit}}+j-i}m_j + \sum_{j=i}^{N_{\text{hit}}}{P_{\text{M}}}^{j-i}m_j\right) + \frac{\beta}{1-P_{\text{M}}}$$

for $1\le i\le N_{\text{hit}}$ and where $\sum_{i=1}^{0} \triangleq 0$.

For $1\le i\le N_{\text{hit}}, 1\le j\le m_j$, let T_{ij} denote the conditional absorption time, conditioned on the start location of the search at the H_0-state $(\bar{i}, j)$. Then

$$T_{ij} = T_{i+1} + (m_i - j + 1)\tau_{\text{P}}$$

with $T_{N_{\text{hit}}+1} \triangleq T_1$.

Once we have the expressions for T_i and T_{ij}, the expression of the absorption time is available:

$$\begin{aligned}v(\boldsymbol{m}) &= \frac{1}{N_{\text{unc}}}\left(\sum_{i=1}^{N_{\text{hit}}} T_i + \sum_{i=1}^{N_{\text{hit}}}\sum_{j=1}^{m_i} T_{ij}\right)\\ &= \frac{1}{N_{\text{unc}}}\sum_{i=1}^{N_{\text{hit}}}\left[T_i + m_i T_{i+1} + \frac{m_i(m_i+1)}{2}\tau_{\text{P}}\right]\\ &= \frac{1}{N_{\text{unc}}}\left[\sum_{i=1}^{N_{\text{hit}}}\left(\frac{\alpha P_{\text{M}}{}^{N_{\text{hit}}-1}}{1-P_{\text{M}}{}^{N_{\text{hit}}}} + \frac{\tau_{\text{P}}}{2}\right)m_i^2\right.\\ &\quad + \sum_{i=1}^{N_{\text{hit}}}\sum_{j=i+1}^{N_{\text{hit}}}\left(\frac{\alpha P_{\text{M}}{}^{j-i-1}}{1-P_{\text{M}}{}^{N_{\text{hit}}}} + \frac{\alpha P_{\text{M}}{}^{N_{\text{hit}}-j+i-1}}{1-P_{\text{M}}{}^{N_{\text{hit}}}}\right)m_i m_j\\ &\quad \left.+ \sum_{i=1}^{N_{\text{hit}}}\left(\frac{\alpha+\beta}{1-P_{\text{M}}} + \frac{\tau_{\text{P}}}{2}\right)m_i + \frac{\beta N_{\text{hit}}}{1-P_{\text{M}}}\right]\\ &\overset{(a)}{=} A\sum_{i=1}^{N_{\text{hit}}} m_i^2 + \sum_{i=1}^{N_{\text{hit}}}\sum_{j=i+1}^{N_{\text{hit}}} B_{ij} m_i m_j + C\\ &= \frac{1}{2}\boldsymbol{m}^T\boldsymbol{H}\boldsymbol{m} + C.\end{aligned}$$

The simplification in (a) uses the constraint

$$\sum_{i=1}^{N_{\text{hit}}} m_i = N_{\text{unc}} - N_{\text{hit}}.$$

The proof is completed. □

In the subsequent analysis, we will allow $\boldsymbol{m}$ in (19) to take noninteger values. In particular, let

$$\mathcal{S}_{\text{c}} = \left\{(x_1, x_2, \ldots, x_{N_{\text{hit}}})\,\middle|\,\sum_{i=1}^{N_{\text{hit}}} x_i = N_{\text{unc}} - N_{\text{hit}}; \forall i, x_i \geq 0\right\} \tag{24}$$

denote the convex hull of $\mathcal{S}_{\text{d}}$ and consider the function $\overline{v}\colon \mathcal{S}_{\text{c}} \to \mathbb{R}$ to be the natural extension of $v\colon \mathcal{S}_{\text{d}} \to \mathbb{R}$. That is, we evaluate $\overline{v}(\boldsymbol{x})$ by simply allowing $v(\cdot)$ in (19) to take the values $\boldsymbol{x} \in \mathcal{S}_{\text{c}}$. Because $\mathcal{S}_{\text{d}} \subset \mathcal{S}_{\text{c}}$, the MAT for any search order π_0 satisfies the following bounds:

$$\min_{\boldsymbol{x}\in\mathcal{S}_{\text{c}}} \overline{v}(\boldsymbol{x}) \leq \mathbb{E}\left\{\mathsf{T}_{\text{ACQ}}(\pi_0)\right\} \leq \max_{\boldsymbol{x}\in\mathcal{S}_{\text{c}}} \overline{v}(\boldsymbol{x}). \tag{25}$$

Also, it can be shown that the set of extreme points of $\mathcal{S}_{\text{c}}$ is given by $\mathcal{E} \subset \mathcal{S}_{\text{d}}$, defined in (15) (see Lemma 1 in Appendix I).

Before delving into the derivations of the bounds in (25) explicitly, we first examine the properties of $\overline{v}(\boldsymbol{x})$. In the next subsection, we use the explicit expression of the absorption time in Theorem 1 to prove important properties of $\overline{v}(\cdot)$.

D. Properties of $\overline{v}(\boldsymbol{x})$

In this subsection, we prove three important properties of $\overline{v}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathcal{S}_{\text{c}}$. These properties are crucial for the development of the forthcoming sections. The three properties are the results of the theorem below.

Theorem 2 (Convexity, Rotational Invariance, and Reversal Invariance): Assume that $P_{\text{M}} < 1$, so that $\overline{v}(\cdot)$ is finite.

1) Function $\overline{v}(\cdot)$ is strictly convex on $\mathcal{S}_{\text{c}}$.
2) $\overline{v}(x_1, x_2, \ldots, x_{N_{\text{hit}}}) = \overline{v}(x_2, x_3, \ldots, x_{N_{\text{hit}}}, x_1)$, $\forall (x_1, x_2, \ldots, x_{N_{\text{hit}}}) \in \mathcal{S}_{\text{c}}$.
3) $\overline{v}(x_1, x_2, \ldots, x_{N_{\text{hit}}}) = \overline{v}(x_{N_{\text{hit}}}, x_{N_{\text{hit}}-1}, \ldots, x_2, x_1)$, $\forall (x_1, x_2, \ldots, x_{N_{\text{hit}}}) \in \mathcal{S}_{\text{c}}$.

Proof:

1) Let any elements $\boldsymbol{x} \in \mathcal{S}_{\text{c}}$ and $\boldsymbol{y} \in \mathcal{S}_{\text{c}}$ be given. For any $\lambda \in (0, 1)$, we want to show that

$$\overline{v}(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) < \lambda\overline{v}(\boldsymbol{x}) + (1-\lambda)\overline{v}(\boldsymbol{y}).$$

Because $(\lambda^2 - \lambda) < 0$ and $\boldsymbol{H}$ is a positive definite matrix (see Appendix II), we conclude that

$$(\lambda^2 - \lambda)(\boldsymbol{x} - \boldsymbol{y})^T\boldsymbol{H}(\boldsymbol{x} - \boldsymbol{y}) < 0.$$

We expand the appropriate terms in the above inequality and have the following results:

$$\begin{aligned}(\lambda^2 - \lambda)\Big(\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{H}\boldsymbol{y}\Big) &< 0\\ \lambda^2\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + 2\lambda(1-\lambda)\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{y} + (1-\lambda)^2\boldsymbol{y}^T\boldsymbol{H}\boldsymbol{y} &<\\ \lambda\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} &+ (1-\lambda)\boldsymbol{y}^T\boldsymbol{H}\boldsymbol{y}\\ \Big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}\Big)^T\boldsymbol{H}\Big(\lambda\boldsymbol{x}^T + (1-\lambda)\boldsymbol{y}\Big) &<\\ \lambda\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} &+ (1-\lambda)\boldsymbol{y}^T\boldsymbol{H}\boldsymbol{y}\\ \overline{v}(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &<\\ \lambda\overline{v}(\boldsymbol{x}) &+ (1-\lambda)\overline{v}(\boldsymbol{y}).\end{aligned}$$

Therefore, $\overline{v}(\cdot)$ is strictly convex on $\mathcal{S}_{\text{c}}$.

2) Let $(x_1, x_2, \ldots, x_{N_{\text{hit}}}) \in \mathcal{S}_{\text{c}}$ be given. Then

$$\begin{aligned}&\overline{v}(x_2, x_3, \ldots, x_{N_{\text{hit}}}, x_1)\\ &= A\sum_{i=1}^{N_{\text{hit}}} x_i^2 + \sum_{i=1}^{N_{\text{hit}}-1}\sum_{j=i+1}^{N_{\text{hit}}-1} B_{ij}x_{i+1}x_{j+1}\\ &\quad + \sum_{i=1}^{N_{\text{hit}}-1} B_{iN_{\text{hit}}}x_{i+1}x_1 + C\\ &\overset{(a)}{=} A\sum_{i=1}^{N_{\text{hit}}} x_i^2 + \sum_{i=1}^{N_{\text{hit}}-1}\sum_{j=i+1}^{N_{\text{hit}}-1} B_{(i+1)(j+1)}x_{i+1}x_{j+1}\\ &\quad + \sum_{i=1}^{N_{\text{hit}}-1} B_{1(i+1)}x_{i+1}x_1 + C\\ &= A\sum_{i=1}^{N_{\text{hit}}} x_i^2 + \sum_{i=2}^{N_{\text{hit}}}\sum_{j=i+1}^{N_{\text{hit}}} B_{ij}x_i x_j + \sum_{j=2}^{N_{\text{hit}}} B_{1j}x_1 x_j + C\\ &= \overline{v}(x_1, x_2, \ldots, x_{N_{\text{hit}}}).\end{aligned}$$

The equality (a) follows from $B_{ij} = B_{(i+1)(j+1)}$ and $B_{iN_{\text{hit}}} = B_{1(i+1)}$.

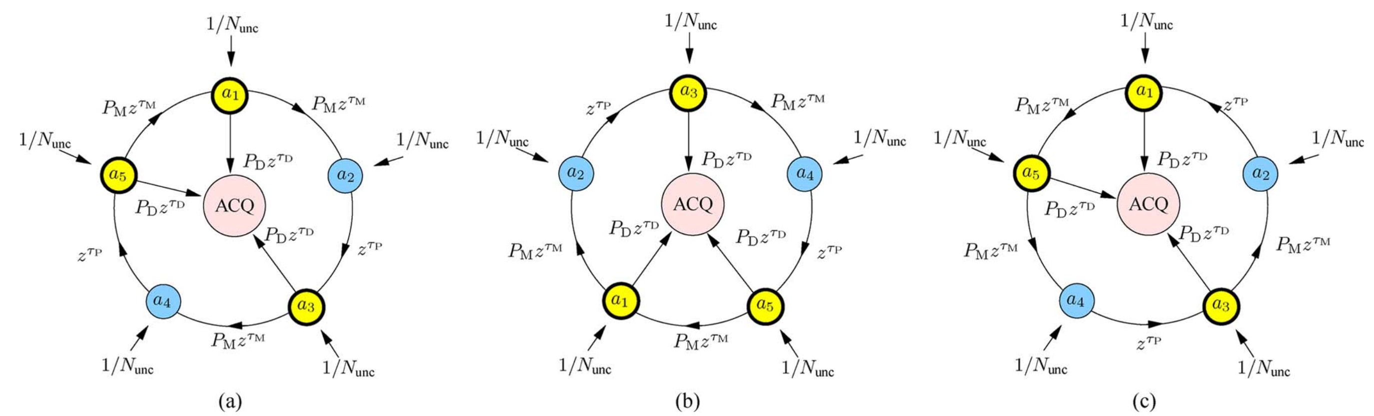

Fig. 7. The three flow diagrams have the same absorption time, but they have different spacing rules. (a) A flow diagram with the spacing rule $(m_1, m_2, \ldots, m_{N_{\text{hit}}})$. (b) A rotated flow diagram with the spacing rule $(m_2, m_3, \ldots, m_{N_{\text{hit}}}, m_1)$. (c) A reversed flow diagram with the spacing rule $(m_{N_{\text{hit}}}, m_{N_{\text{hit}}-1}, \ldots, m_1)$. To simplify the drawing, we show the case when $N_{\text{unc}} = 5$, $N_{\text{hit}} = 3$, and $(m_1, m_2, m_3) = (1, 1, 0)$.

3) Let $(x_1, x_2, \ldots, x_{N_{\text{hit}}}) \in \mathcal{S}_{\text{c}}$ be given. Then

$$\begin{aligned}
&\overline{v}(x_{N_{\text{hit}}}, x_{N_{\text{hit}}-1}, \ldots, x_2, x_1) \\
&= A \sum_{i=1}^{N_{\text{hit}}} x_i^2 + \sum_{i=1}^{N_{\text{hit}}} \sum_{j=i+1}^{N_{\text{hit}}} B_{ij} x_{N_{\text{hit}}-i+1} x_{N_{\text{hit}}-j+1} + C \\
&\overset{\text{(a)}}{=} A \sum_{i=1}^{N_{\text{hit}}} x_i^2 \\
&\quad + \sum_{i=1}^{N_{\text{hit}}} \sum_{j=i+1}^{N_{\text{hit}}} B_{(N_{\text{hit}}-j+1)(N_{\text{hit}}-i+1)} x_{N_{\text{hit}}-i+1} x_{N_{\text{hit}}-j+1} \\
&\quad + C \\
&= A \sum_{i=1}^{N_{\text{hit}}} x_i^2 + \sum_{i=1}^{N_{\text{hit}}} \sum_{j=i+1}^{N_{\text{hit}}} B_{ij} x_i x_j + C \\
&= \overline{v}(x_1, x_2, \ldots, x_{N_{\text{hit}}}).
\end{aligned}$$

The equality (a) follows from

$$B_{ij} = B_{(N_{\text{hit}}-j+1)(N_{\text{hit}}-i+1)}.$$

That completes the proof. □

We provide an interpretation of the second and third properties associated with nonpreferential flow diagrams. The second property states that the absorption time is invariant when every state in the flow diagram is rotated to the left. Applying the second property to the flow diagram several times, we can show that the absorption time is also invariant when the flow diagram is rotated to the right. Thus, the absorption time is rotationally invariant. The third property states that the absorption time is invariant when the flow diagram is viewed in a reverse direction. See Fig. 7 for an illustration.

The rotational and reversal invariance implies that the search orders $\pi^{N_{\text{J}}}$ and $\pi^{N_{\text{unc}}-N_{\text{J}}}$ give the same MAT for the following reason. The search order $\pi^{N_{\text{J}}}$ tests cells in the reverse order of the search order $\pi^{N_{\text{unc}}-N_{\text{J}}}$. Thus, if

$$\boldsymbol{s}(\pi^{N_{\text{J}}}, b) = (m_1, m_2, \ldots, m_{N_{\text{hit}}})$$

for some $b \in \mathcal{U}$ is the spacing rule associated with the description $(\pi^{N_{\text{J}}}, b)$ then

$$\begin{aligned}
&\boldsymbol{s}(\pi^{N_{\text{unc}}-N_{\text{J}}}, b) \\
&\quad = \begin{cases} (m_{N_{\text{hit}}}, \ldots, m_2, m_1), & 1 \in \mathcal{H}_{\text{hit}}(b) \\ (m_{N_{\text{hit}}-1}, m_{N_{\text{hit}}-2}, \ldots, m_1, m_{N_{\text{hit}}}), & \text{otherwise} \end{cases}
\end{aligned}$$

is the spacing rule associated with the description $(\pi^{N_{\text{unc}}-N_{\text{J}}}, b)$. The spacing rules $(m_1, m_2, \ldots, m_{N_{\text{hit}}})$, $(m_{N_{\text{hit}}}, \ldots, m_2, m_1)$, and $(m_{N_{\text{hit}}-1}, m_{N_{\text{hit}}-2}, \ldots, m_1, m_{N_{\text{hit}}})$ result in the same absorption time by the rotational and reversal invariance. Hence the MATs associated with the search orders $\pi^{N_{\text{J}}}$ and $\pi^{N_{\text{unc}}-N_{\text{J}}}$ are equal

$$\mathbb{E}\left\{\mathsf{T}_{\text{ACQ}}(\pi^{N_{\text{J}}})\right\} = \mathbb{E}\left\{\mathsf{T}_{\text{ACQ}}(\pi^{N_{\text{unc}}-N_{\text{J}}})\right\}. \tag{26}$$

Remark 2: Although $\overline{v}(\cdot)$ is convex on $\mathcal{S}_{\text{c}}$, it is not *Schur* convex. Consider a simple counterexample, in which $N_{\text{unc}} = 14$, $N_{\text{hit}} = 4$, $P_{\text{D}} = P_{\text{M}} = \frac{1}{2}$, $\tau_{\text{D}} = \tau_{\text{M}} = 1$, and $\tau_{\text{P}} = 10$. Then, $\overline{v}(1, 2, 3, 4) = 51\frac{19}{21}$, $\overline{v}(1, 3, 4, 2) = 40\frac{4}{7}$, and $\overline{v}(1, 2, 3, 4) \neq \overline{v}(1, 3, 4, 2)$. Because $\overline{v}(\cdot)$ is not permutational invariant, it is not *Schur* convex. Therefore, optimizing $\overline{v}(\cdot)$ is not a straightforward task. In the next section, we will use the explicit expression $\overline{v}(\cdot)$ and its properties to minimize the absorption time and bound the minimum MAT.

IV. The Minimum MAT

In this section, we find the upper and lower bounds for the minimum MAT

$$T_{\min} \triangleq \min_{\pi \in \mathcal{P}} \mathbb{E}\left\{\mathsf{T}_{\text{ACQ}}(\pi)\right\}.$$

We will show that for certain values of N_{hit}, there *exists* a search order that achieves the lower bound. Furthermore, we will obtain a "near-optimal" search order that results in the MAT reasonably close to the minimum one. The lower bound of $T_{\min}$ is given in the following theorem.

Theorem 3 (Minimum MAT): The optimal mean acquisition time $T_{\min}$ satisfies

$$T_{\min}^{\text{L}} \leq T_{\min} \tag{27}$$

where $T_{\min}^{\mathrm{L}}$ is defined to be

$$T_{\min}^{\mathrm{L}} \triangleq \left(\frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1\right)\left(\frac{1+P_{\mathrm{M}}}{1-P_{\mathrm{M}}}\right)\frac{\tau_{\mathrm{P}}}{2} + \frac{P_{\mathrm{M}}}{1-P_{\mathrm{M}}}\tau_{\mathrm{M}} + \tau_{\mathrm{D}}. \tag{28}$$

Moreover, the equality in (27) is achieved if and only if $N_{\mathrm{hit}} = 1$ or $N_{\mathrm{hit}} = N_{\mathrm{unc}}$.

Proof: $T_{\min}$ is lower-bounded by

$$\begin{aligned} T_{\min} &= \frac{1}{N_{\mathrm{unc}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{unc}}} v(\boldsymbol{s}(\pi, b)) \\ &\geq \frac{1}{N_{\mathrm{unc}}} \sum_{b=1}^{N_{\mathrm{unc}}} \min_{\pi \in \mathcal{P}} v(\boldsymbol{s}(\pi, b)) \\ &\geq \frac{1}{N_{\mathrm{unc}}} \sum_{b=1}^{N_{\mathrm{unc}}} \min_{(\pi, i) \in \mathcal{D}} v(\boldsymbol{s}(\pi, i)) \\ &= \min_{\boldsymbol{m} \in \mathcal{S}_{\mathrm{d}}} v(\boldsymbol{m}) \\ &\geq \min_{\boldsymbol{x} \in \mathcal{S}_{\mathrm{c}}} \overline{v}(\boldsymbol{x}) \\ &\overset{(a)}{=} T_{\min}^{\mathrm{L}}. \end{aligned} \tag{29}$$

The equality (a) follows from part two of Lemma 3 in Appendix III, which shows that

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathcal{S}_{\mathrm{c}}} \overline{v}(\boldsymbol{x}) &= \overline{v}\left(\frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1, \frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1, \ldots, \frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1\right) \\ &= T_{\min}^{\mathrm{L}}. \end{aligned} \tag{30}$$

Therefore, we have the bound $T_{\min} \geq T_{\min}^{\mathrm{L}}$.

Now, we show that the equality in (27) is achieved if and only if $N_{\mathrm{hit}} = 1$ or $N_{\mathrm{hit}} = N_{\mathrm{unc}}$.

Assume that there is only one in-phase cell ($N_{\mathrm{hit}} = 1$). Then, for any description $(\pi, b) \in \mathcal{D}$, the spacing rule satisfies $\boldsymbol{s}(\pi, b) = (N_{\mathrm{unc}} - 1)$ and the absorption time $v(\boldsymbol{s}(\pi, b)) = v(N_{\mathrm{unc}} - 1)$ is a constant for all $b \in \mathcal{U}$.[11] The optimal MAT satisfies

$$\begin{aligned} T_{\min} &= \frac{1}{N_{\mathrm{unc}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{unc}}} v(\boldsymbol{s}(\pi, b)) \\ &= \frac{1}{N_{\mathrm{unc}}} \min_{\pi \in \mathcal{P}} \sum_{b=1}^{N_{\mathrm{unc}}} v(N_{\mathrm{unc}} - 1) \\ &= v(N_{\mathrm{unc}} - 1) \\ &= T_{\min}^{\mathrm{L}}. \end{aligned}$$

Next, assume that all cells are in-phase cells ($N_{\mathrm{hit}} = N_{\mathrm{unc}}$). Then, for any description $(\pi, b) \in \mathcal{D}$, the spacing rule satisfies $\boldsymbol{s}(\pi, b) = (0, 0, \ldots, 0)$ and the absorption time $v(\boldsymbol{s}(\pi, b)) = v(0, 0, \ldots, 0)$ is again a constant for all $b \in \mathcal{U}$. Using similar steps to the case for $N_{\mathrm{hit}} = 1$ shows that the optimal MAT satisfies $T_{\min} = T_{\min}^{\mathrm{L}}$. Therefore, if $N_{\mathrm{hit}} = 1$ or $N_{\mathrm{hit}} = N_{\mathrm{unc}}$, the equality in (27) is achieved.

[11]When $N_{\mathrm{hit}} = 1$, the spacing rule contains only one element.

We will give a contrapositive proof to show that the equality in (27) implies $N_{\mathrm{hit}} = 1$ or $N_{\mathrm{hit}} = N_{\mathrm{unc}}$. Assume that $N_{\mathrm{hit}} \neq 1$ and $N_{\mathrm{hit}} \neq N_{\mathrm{unc}}$. Lemma 4 in Appendix IV implies that for the optimal search order

$$\pi^* \triangleq \arg\min_{\pi \in \mathcal{P}} \mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi)\right\}$$

there exists $b_0 \in \mathcal{U}$ such that

$$\boldsymbol{s}(\pi^*, b_0) \neq \left(\frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1, \frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1, \ldots, \frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1\right). \tag{31}$$

Part one of Lemma 3 in Appendix III shows that the right-hand side of (31) is the *unique* minimizer of $\overline{v}(\cdot)$. As a result, the absorption time $v(\boldsymbol{s}(\pi^*, b_0))$ satisfies the following strict inequality:

$$\begin{aligned} v(\boldsymbol{s}(\pi^*, b_0)) &> \overline{v}\left(\frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1, \frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1, \ldots, \frac{N_{\mathrm{unc}}}{N_{\mathrm{hit}}} - 1\right) \\ &= \min_{\boldsymbol{x} \in \mathcal{S}_{\mathrm{c}}} \overline{v}(\boldsymbol{x}). \end{aligned} \tag{32}$$

Then, the minimum MAT is strictly greater than its lower bound

$$\begin{aligned} T_{\min} &= \mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi^*)\right\} \\ &= \frac{1}{N_{\mathrm{unc}}}\left[v(\boldsymbol{s}(\pi^*, b_0)) + \sum_{\substack{b=1 \\ b \neq b_0}}^{N_{\mathrm{unc}}} v(\boldsymbol{s}(\pi^*, b))\right] \\ &\overset{(a)}{>} \frac{1}{N_{\mathrm{unc}}}\left[\min_{\boldsymbol{x} \in \mathcal{S}_{\mathrm{c}}} \overline{v}(\boldsymbol{x}) + \sum_{\substack{b=1 \\ b \neq b_0}}^{N_{\mathrm{unc}}} v(\boldsymbol{s}(\pi^*, b))\right] \\ &\geq \frac{1}{N_{\mathrm{unc}}}\left[\min_{\boldsymbol{x} \in \mathcal{S}_{\mathrm{c}}} \overline{v}(\boldsymbol{x}) + \sum_{\substack{b=1 \\ b \neq b_0}}^{N_{\mathrm{unc}}} \min_{\boldsymbol{x} \in \mathcal{S}_{\mathrm{c}}} \overline{v}(\boldsymbol{x})\right] \\ &= \min_{\boldsymbol{x} \in \mathcal{S}_{\mathrm{c}}} \overline{v}(\boldsymbol{x}) \\ &= T_{\min}^{\mathrm{L}}. \end{aligned} \tag{33}$$

The inequality (a) follows from (32). Therefore, the equality in (27) is not achieved. This completes the proof. □

Next, we introduce the concept of the η-optimal search order and η-optimal spacing rule. We then show that if N_{hit} and N_{unc} are relatively prime, the MAT $\mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi^{N_{\mathrm{hit}}})\right\}$ achieved by the search order $\pi^{N_{\mathrm{hit}}}$ is η-optimal.

Definition 4 (η-Optimal Search Order): Let $\eta(N_{\mathrm{hit}}, N_{\mathrm{unc}})$ be a function only of N_{hit} and N_{unc}. A search order π is η-optimal if

$$\frac{\mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi)\right\} - T_{\min}^{\mathrm{L}}}{T_{\min}^{\mathrm{L}}} \leq \eta(N_{\mathrm{hit}}, N_{\mathrm{unc}}) \tag{34}$$

and $\eta(N_{\mathrm{hit}}, N_{\mathrm{unc}}) \to 0$ as the ratio $N_{\mathrm{hit}}/N_{\mathrm{unc}} \to 0$.

Note that if π is an η-optimal search order, then

$$\begin{aligned} \frac{\mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi)\right\} - \mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi^*)\right\}}{\mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi^*)\right\}} &\leq \frac{\mathbb{E}\left\{\mathsf{T}_{\mathrm{ACQ}}(\pi)\right\} - T_{\min}^{\mathrm{L}}}{T_{\min}^{\mathrm{L}}} \\ &\leq \eta(N_{\mathrm{hit}}, N_{\mathrm{unc}}) \end{aligned}$$

where π^* is an optimal search order that minimizes the MAT and $\eta(N_{\mathrm{hit}}, N_{\mathrm{unc}}) \to 0$ as the ratio $N_{\mathrm{hit}}/N_{\mathrm{unc}} \to 0$. Therefore, an η-optimal search order can achieve a MAT arbitrarily

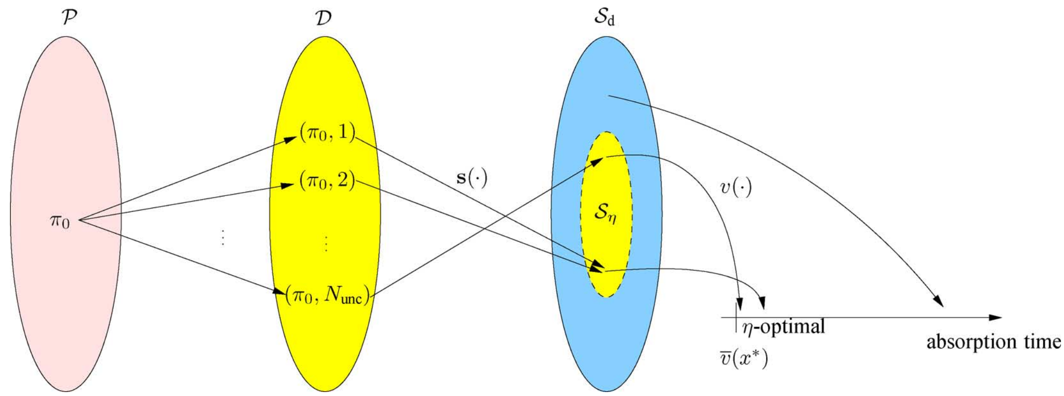

Fig. 8. The search order π_0 is η-optimal, because the spacing rules $\boldsymbol{s}(\pi_0, 1), \boldsymbol{s}(\pi_0, 2), \ldots, \boldsymbol{s}(\pi_0, N_{\rm unc})$ are members of an η-optimal subset $\mathcal{S}_\eta \subset \mathcal{S}_{\rm d}$.

close to that of the optimal search order as the ratio $N_{\rm hit}/N_{\rm unc}$ approaches zero.

Recall that signal acquisition is a challenging task when the total number of in-phase cells is significantly smaller than the total number of cells. In this situation, which is typical for UWB systems, the acquisition time can be intolerably long and there are many scenarios and applications that necessitate the use of faster acquisition techniques. Note that $N_{\rm hit}/N_{\rm unc}$ is small and approaches zero as the demand for faster acquisition is intensified. Our goal is to find η-optimal solutions, because they are almost as good as the optimal in a situation where rapid acquisition is of upmost importance.

Definition 5 (η-Optimal Subset of Spacing Rules): Let $\eta(N_{\rm hit}, N_{\rm unc})$ be a function only of $N_{\rm hit}$ and $N_{\rm unc}$. A subset $\mathcal{S}_\eta \subset \mathcal{S}_{\rm d}$ is η-optimal, if for every $\boldsymbol{m} \in \mathcal{S}_\eta$

$$\frac{v(\boldsymbol{m}) - T^{\rm L}_{\rm min}}{T^{\rm L}_{\rm min}} \leq \eta(N_{\rm hit}, N_{\rm unc}) \tag{35}$$

and $\eta(N_{\rm hit}, N_{\rm unc}) \to 0$ as the ratio $N_{\rm hit}/N_{\rm unc} \to 0$.

The relationship between the η-optimal search order and the η-optimal spacing rules (see Fig. 8) is established by the η-Isometry Property (Lemma 5 in Appendix V).

In particular, the lemma states that if the search order π satisfies

$$\boldsymbol{s}(\pi, b) \in \mathcal{S}_\eta, \qquad b = 1, 2, \ldots, N_{\rm unc}$$

for some η-optimal subset $\mathcal{S}_\eta \subset \mathcal{S}_{\rm d}$, then π is η-optimal. In the next theorem, we use this relationship to prove that the search order $\pi^{N_{\rm hit}}$ is η-optimal with $\eta = \left(\frac{2N_{\rm hit}}{N_{\rm unc} - N_{\rm hit}}\right)$.

Theorem 4 (Near-Optimality): If $N_{\rm hit}$ and $N_{\rm unc}$ are relatively prime, then the search order $\pi^{N_{\rm hit}}$ is η-optimal with

$$\eta = \frac{2N_{\rm hit}}{N_{\rm unc} - N_{\rm hit}}.$$

Furthermore, we have the inequalities

$$\begin{aligned} T^{\rm L}_{\rm min} &\leq \mathbb{E}\left\{\mathsf{T}_{\rm ACQ}(\pi^*)\right\} \\ &\leq \mathbb{E}\left\{\mathsf{T}_{\rm ACQ}(\pi^{N_{\rm hit}})\right\} \leq \left(1 + \frac{2N_{\rm hit}}{N_{\rm unc} - N_{\rm hit}}\right) T^{\rm L}_{\rm min} \end{aligned} \tag{36}$$

in which π^* denotes an optimal search order that minimizes the MAT.

Proof: Let

$$\mathcal{R} \triangleq \left\{ (m_1, m_2, \ldots, m_{N_{\rm hit}}) \middle| \sum_{i=1}^{N_{\rm hit}} m_i = N_{\rm unc} - N_{\rm hit}; \right. \\ \left. \forall i, m_i \in \mathbb{Z} \text{ and } 0 \leq m_i \leq \left\lfloor \frac{N_{\rm unc}}{N_{\rm hit}} \right\rfloor \right\} \tag{37}$$

be a subset of $\mathcal{S}_{\rm d}$. Lemma 6 in Appendix VI shows that $\mathcal{R}$ is η-optimal with $\eta = \left(\frac{2N_{\rm hit}}{N_{\rm unc} - N_{\rm hit}}\right)$. Lemma 7 in Appendix VII shows that

$$\boldsymbol{s}(\pi^{N_{\rm hit}}, b) \in \mathcal{R}, \qquad \text{for all } b = 1, 2, \ldots, N_{\rm hit}.$$

Therefore, by Lemma 5, the search order $\pi^{N_{\rm hit}}$ is η-optimal with $\eta = \left(\frac{2N_{\rm hit}}{N_{\rm unc} - N_{\rm hit}}\right)$.

The first inequality in (36) follows from a lower bound of the minimum MAT in (27). The second inequality follows from the definition of an optimal search order π^*. The third inequality follows from the definition of η-optimality. That completes the proof. $\square$

In the next section, we derive the search orders that result in the maximum MAT.

V. The Maximum MAT

In this section, we show that the CSS and the FSSS with the step size $N_{\rm unc} - 1$ both yield the maximum MAT, and thus should be avoided for signal acquisition in multipath environments.

Theorem 5 (Maximum MAT):

1) The expression for the maximum MAT is given by

$$\begin{aligned} T_{\rm max} &\triangleq \max_{\pi \in \mathcal{P}} \mathbb{E}\left\{\mathsf{T}_{\rm ACQ}(\pi)\right\} \\ &= \frac{(N_{\rm unc} - N_{\rm hit})^2}{N_{\rm unc}} \cdot \left(\frac{1 + {P_{\rm M}}^{N_{\rm hit}}}{1 - {P_{\rm M}}^{N_{\rm hit}}}\right) \frac{\tau_{\rm P}}{2} \\ &\quad + \left(1 - \frac{N_{\rm hit}}{N_{\rm unc}}\right) \cdot \left(\frac{1 + P_{\rm M}}{1 - P_{\rm M}}\right) \frac{\tau_{\rm P}}{2} \\ &\quad + \frac{P_{\rm M}}{1 - P_{\rm M}} \tau_{\rm M} + \tau_{\rm D}. \end{aligned} \tag{38}$$

If the receiver uses the CSS π^1 or the FSSS $\pi^{N_{\text{unc}}-1}$, it will result in the maximum MAT.

2) If the number N_{hit} of in-phase cells satisfies $2 \le N_{\text{hit}} \le N_{\text{unc}} - 2$ and the receiver's MAT is equal to $T_{\max}$ in (38), then the receiver must have used the CSS π^1 or the FSSS $\pi^{N_{\text{unc}}-1}$.

Proof:

1) The search orders π^1 and $\pi^{N_{\text{unc}}-1}$ correspond to the N_{unc}-tuples $[1, 2, 3, \ldots, N_{\text{unc}}]$ and $[1, N_{\text{unc}}, N_{\text{unc}} - 1, \ldots, 3, 2]$, respectively. For any $b \in \mathcal{U}$, careful thought will reveal that the spacing rules $\boldsymbol{s}(\pi^1, b)$ and $\boldsymbol{s}(\pi^{N_{\text{unc}}-1}, b)$ satisfy

$$\boldsymbol{s}(\pi^1, b) \in \mathcal{E} \tag{39}$$

$$\boldsymbol{s}(\pi^{N_{\text{unc}}-1}, b) \in \mathcal{E} \tag{40}$$

where $\mathcal{E}$ is defined in (15). As a result

$$\begin{aligned}\mathbb{E}\left\{\mathsf{T}_{\text{ACQ}}(\pi^1)\right\} &= \frac{1}{N_{\text{unc}}} \sum_{b=1}^{N_{\text{unc}}} v(\boldsymbol{s}(\pi^1, b)) \\ &\overset{\text{(a)}}{=} \frac{1}{N_{\text{unc}}} \sum_{b=1}^{N_{\text{unc}}} \max_{\boldsymbol{m} \in \mathcal{S}_{\text{d}}} v(\boldsymbol{m}) \\ &= \max_{\boldsymbol{m} \in \mathcal{S}_{\text{d}}} v(\boldsymbol{m}) \\ &\overset{\text{(b)}}{=} T_{\max}.\end{aligned}$$

The equality (a) follows from (39) and part one of Lemma 8 of Appendix VIII, which shows that elements of $\mathcal{E}$ are solutions of $\max_{\boldsymbol{m} \in \mathcal{S}_{\text{d}}} v(\boldsymbol{m})$. The equality (b) follows from part two of Lemma 8, which gives the explicit closed-form expression for the maximum absorption time. Similar steps also reveal that $\mathbb{E}\left\{\mathsf{T}_{\text{ACQ}}(\pi^{N_{\text{unc}}-1})\right\} = T_{\max}$. Therefore, the search orders π^1 and $\pi^{N_{\text{unc}}-1}$ maximize the MAT.

2) Let N_{hit} satisfy $2 \le N_{\text{hit}} \le N_{\text{unc}} - 2$. Let π_{w} be a search order that maximizes the MAT. The Strong Clustering Property (Lemma 10 of Appendix IX) implies that $\{1, 2\} = \{\pi_{\text{w}}(k), \pi_{\text{w}}(k \oplus 1)\}$ for some $k \in \mathcal{U}$. Note that $\pi_{\text{w}}(1) = 1$ by the definition of a search order.
If $\pi_{\text{w}}(1) = \pi_{\text{w}}(k)$, then $\pi_{\text{w}}(2) = \pi_{\text{w}}(k \oplus 1) = 2$. On the other hand, if $\pi_{\text{w}}(1) = \pi_{\text{w}}(k \oplus 1)$, then $\pi_{\text{w}}(N_{\text{unc}}) = \pi_{\text{w}}(k) = 2$. Therefore, $\pi_{\text{w}}(2) = 2$ or $\pi_{\text{w}}(N_{\text{unc}}) = 2$. We consider these two cases separately.

- $\pi_{\text{w}}(2) = 2$.
The Strong Clustering Property (Lemma 10) implies that $\{2, 3\} = \{\pi_{\text{w}}(i), \pi_{\text{w}}(i \oplus 1)\}$ for some $i \in \mathcal{U}$. If $\pi_{\text{w}}(2) = \pi_{\text{w}}(i \oplus 1)$, then $\pi_{\text{w}}(1) = \pi_{\text{w}}(i) = 3$, and we have a contradiction: $1 = \pi_{\text{w}}(1) = 3$. Therefore, $\pi_{\text{w}}(2) = \pi_{\text{w}}(i)$ and $\pi_{\text{w}}(3) = \pi_{\text{w}}(i \oplus 1) = 3$. A similar argument shows that
$$\pi_{\text{w}}(4) = 4,\ \pi_{\text{w}}(5) = 5,\ \ldots,\ \pi_{\text{w}}(N_{\text{unc}}) = N_{\text{unc}}.$$
Therefore, π_{w} is the CSS: $\pi_{\text{w}} = \pi^1$.
- $\pi_{\text{w}}(N_{\text{unc}}) = 2$.
The Strong Clustering Property (Lemma 10) implies that $\{2, 3\} = \{\pi_{\text{w}}(j), \pi_{\text{w}}(j \oplus 1)\}$ for some $j \in \mathcal{U}$. If $\pi_{\text{w}}(N_{\text{unc}}) = \pi_{\text{w}}(j)$, then $\pi_{\text{w}}(1) = \pi_{\text{w}}(j \oplus 1) = 3$, and we have a contradiction: $1 = \pi_{\text{w}}(1) = 3$. Therefore, $\pi_{\text{w}}(N_{\text{unc}}) = \pi_{\text{w}}(j \oplus 1)$ and $\pi_{\text{w}}(N_{\text{unc}} - 1) = \pi_{\text{w}}(j) = 3$. A similar argument shows that
$$\pi_{\text{w}}(N_{\text{unc}} - 2) = 4,$$
$$\pi_{\text{w}}(N_{\text{unc}} - 3) = 5,\ \ldots,\ \pi_{\text{w}}(2) = N_{\text{unc}}.$$
Therefore, π_{w} is the FSSS with the step size $N_{\text{unc}} - 1$: $\pi_{\text{w}} = \pi^{N_{\text{unc}}-1}$.

These two cases imply that π_{w} is π^1 or $\pi^{N_{\text{unc}}-1}$. That completes the proof. □

Remark 3: Before we end this section, we note that the range $2 \le N_{\text{hit}} \le N_{\text{unc}} - 2$ in Theorem 5 cannot be expanded. In particular, for $N_{\text{hit}} \in \{1, N_{\text{unc}} - 1, N_{\text{unc}}\}$, the search order π_{w} that maximizes the MAT is not necessarily the search order π^1 or $\pi^{N_{\text{unc}}-1}$. This is trivial when $N_{\text{hit}} = 1$ or $N_{\text{hit}} = N_{\text{unc}}$ because every search order results in the same MAT. For $N_{\text{hit}} = N_{\text{unc}} - 1$, we provide a simple counterexample, in which $N_{\text{hit}} = 3$, $N_{\text{unc}} = 4$, and π_{w} corresponds to the 4-tuple $[1, 3, 4, 2]$. As shown in Fig. 9, the corresponding flow diagram for each B gives

$$\begin{aligned}\boldsymbol{s}(\pi_{\text{w}}, 1) &= (0, 1, 0) \\ \boldsymbol{s}(\pi_{\text{w}}, 2) &= (0, 0, 1) \\ \boldsymbol{s}(\pi_{\text{w}}, 3) &= (0, 0, 1) \text{ and} \\ \boldsymbol{s}(\pi_{\text{w}}, 4) &= (1, 0, 0).\end{aligned}$$

Note that Lemma 8 implies that these spacing rules result in the maximum absorption time, and thus

$$\begin{aligned}v(\boldsymbol{s}(\pi_{\text{w}}, 1)) &= v(\boldsymbol{s}(\pi_{\text{w}}, 2)) \\ &= v(\boldsymbol{s}(\pi_{\text{w}}, 3)) = v(\boldsymbol{s}(\pi_{\text{w}}, 4)) = T_{\max}.\end{aligned}$$

As a result, π_{w} yields the maximum MAT. Evidently, this search order π_{w} is not the search order π^1 or $\pi^{N_{\text{unc}}-1}$.

In typical scenarios, N_{hit} is in the range $2 \le N_{\text{hit}} \le N_{\text{unc}} - 2$. In these scenarios, the receiver exhibits the maximum MAT if and only if it uses the CSS or the FSSS with the step size $N_{\text{unc}} - 1$. Therefore, the receiver can immediately improve the MAT by choosing another search order, other than the worst search orders π^1 and $\pi^{N_{\text{unc}}-1}$.

VI. Conclusion

This paper provides a methodology for exploiting multipath, typically considered deleterious for efficient communications, to aid the sequence acquisition. We consider a class of serial search strategies and model each search procedure by a *non-preferential* flow diagram, containing N_{unc} total cells and N_{hit} in-phase cells.

We first demonstrate the difficulty associated with direct optimization of the MAT over a set of descriptions. This difficulty is alleviated by transforming the *descriptions* into the *spacing rules* and deriving the expression of the MAT as an explicit function of the spacing rule. In this new framework, finding the fundamental limits of the achievable MATs is equivalent to solving convex optimization problems. Solutions to those optimization problems give insights into the minimum and maximum MATs.

We derive a lower bound and an upper bound on the minimum MAT. The lower bound is achieved with equality if and only if

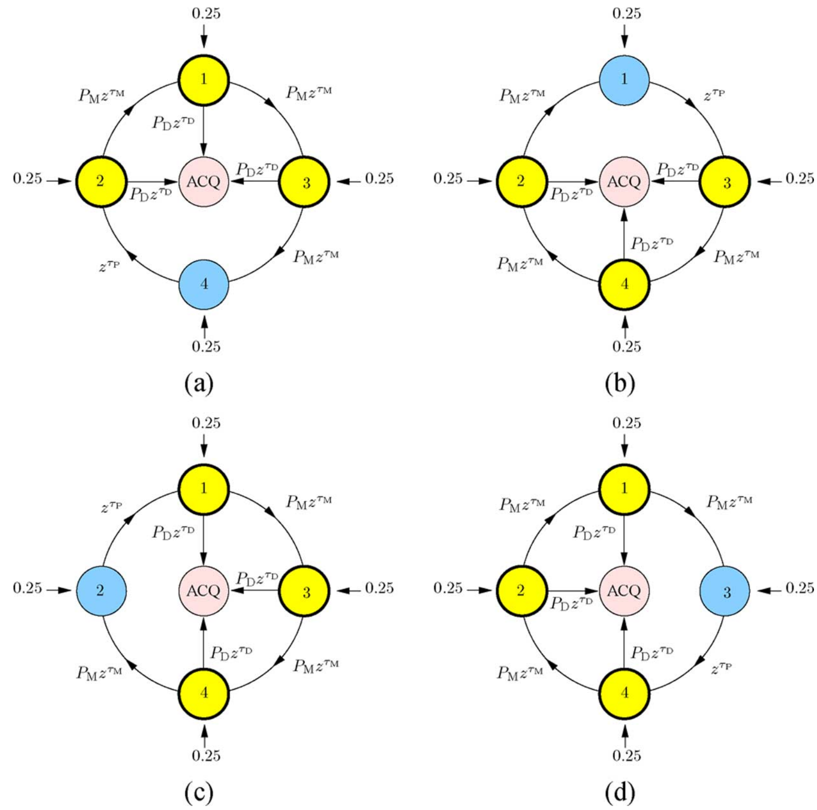

Fig. 9. When $N_{\rm hit} = 3$ and $N_{\rm unc} = 4$, the search order $[1, 3, 4, 2]$ maximizes the MAT because the H_1-states in $\mathcal{H}_{\rm hit}(b)$ are *clustering* for every location $\mathsf{B} = b$ of the first in-phase cell. (a) $\mathsf{B} = 1$ and $\mathcal{H}_{\rm hit}(1) = \{1, 2, 3\}$. (b) $\mathsf{B} = 2$ and $\mathcal{H}_{\rm hit}(2) = \{2, 3, 4\}$. (c) $\mathsf{B} = 3$ and $\mathcal{H}_{\rm hit}(3) = \{3, 4, 1\}$. (d) $\mathsf{B} = 4$ and $\mathcal{H}_{\rm hit}(4) = \{4, 1, 2\}$.

there is one in-phase cell ($N_{\rm hit} = 1$) or there are $N_{\rm unc}$ in-phase cells ($N_{\rm hit} = N_{\rm unc}$). We introduce the notion of η-optimality and prove that the fixed-step serial search (FSSS) with the step size $N_{\rm hit}$ is η-optimal. As a consequence, the FSSS with the step size $N_{\rm hit}$ can be effectively used to achieve the near-optimal MAT in wide-bandwidth transmission systems operating in dense multipath channels.

We also investigate the search orders that result in the maximum MAT. It turns out that the conventional serial search (CSS) and the FSSS with step size $N_{\rm unc} - 1$ exhibit the maximum MAT. For a typical scenario with $2 \le N_{\rm hit} \le N_{\rm unc} - 2$, we further show that only those two search orders result in the maximum MAT. Therefore, the receiver can immediately improve the MAT by avoiding the CSS or the FSSS with the step size $N_{\rm unc} - 1$. Our results are valid for all SNR values, detection layer decision rules, and fading distributions.

We note that the η-optimal search order $\pi^{N_{\rm hit}}$ requires knowledge of the multipath dispersion interval. When the exact dispersion interval is unknown or changing, a receiver may employ the FSSS with the step size $N_{\rm hit}^{\rm low}$ as a conservative choice, where $N_{\rm hit}^{\rm low}$ denotes a lower bound on $N_{\rm hit}$. In addition, we note that the proofs of the lower bound (Theorem 3) and the upper bound ((38), Theorem 5) on the MATs do not require propagation paths to arrive in a single cluster. Hence, these bounds are valid for an environment in which multiple clusters of propagation paths are observed at the receiver. We have deliberately focused our attention in this paper on search-layer issues and have abstracted details of the decision layer into a few parameters $P_{\rm D}$, $P_{\rm M}$, $\tau_{\rm D}$, $\tau_{\rm M}$, and $\tau_{\rm P}$. Future extensions of this work include a study of the implication of various fading statistics as well as a study of detection-layer strategies such as a MAX/TC criterion, the optimal decision rules, and multipath combining methods, in conjunction with the optimal and near-optimal search strategies.

APPENDIX I
EXTREME POINTS

This appendix shows an important relationship between the set $\mathcal{E}$ of spacing rules associated with the CSS and the convex hull $\mathcal{S}_{\rm c}$ of the spacing rules. Sets $\mathcal{E}$ and $\mathcal{S}_{\rm c}$ are defined in (15) and (24), respectively.

Lemma 1 (Extreme Points): Set $\mathcal{E}$ is the set of extreme points of $\mathcal{S}_{\rm c}$.

Proof: Let any index $1 \le i \le N_{\rm hit}$ be given. There are $N_{\rm hit}$ linearly independent constraints of $\mathcal{S}_{\rm c}$ (listed below) that are active at $\boldsymbol{m}^{(i)} \in \mathcal{E}$

$$\sum_{j=1}^{N_{\rm hit}} m_j^{(i)} = N_{\rm unc} - N_{\rm hit} \quad \text{(one constraint)} \tag{41}$$

$$m_j^{(i)} = 0, \qquad \text{for } j = 1, 2, \ldots, N_{\rm hit} \text{ and } j \ne i \quad (N_{\rm hit} - 1 \text{ constraints}). \tag{42}$$

Thus, $\boldsymbol{m}^{(i)}$ is an extreme point, and $\mathcal{E} \subset \mathcal{E}_{\rm x}$, where $\mathcal{E}_{\rm x}$ denotes the set of extreme points of $\mathcal{S}_{\rm c}$.

Conversely, let any extreme point $\boldsymbol{m} \in \mathcal{E}_{\rm x}$ be given. Then, $N_{\rm hit}$ linearly independent constraints of $\mathcal{S}_{\rm c}$ are active at $\boldsymbol{m}$. By definition of $\mathcal{S}_{\rm c}$, the active constraints must be the following:

$$\sum_{j=1}^{N_{\rm hit}} m_j = N_{\rm unc} - N_{\rm hit} \quad \text{(one constraint)} \tag{43}$$

$$m_j = 0, \qquad \text{for } j = 1, 2, \ldots, N_{\rm hit} \text{ and } j \neq k \quad (N_{\rm hit} - 1 \text{ constraints}) \tag{44}$$

for some $1 \leq k \leq N_{\rm hit}$. Thus, $\boldsymbol{m} = \boldsymbol{m}^{(k)} \in \mathcal{E}$, and $\mathcal{E}_{\rm x} \subset \mathcal{E}$. Therefore, $\mathcal{E} = \mathcal{E}_{\rm x}$. That completes the proof. □

Appendix II
Positive Definiteness of the Hessian Matrix $\boldsymbol{H}$

The goal of this appendix is to show that an $N_{\rm hit} \times N_{\rm hit}$ matrix

$$\boldsymbol{H} \triangleq \frac{\tau_{\rm P}}{N_{\rm unc}\left(1 - {P_{\rm M}}^{N_{\rm hit}}\right)} \left[{P_{\rm M}}^{N_{\rm hit}-|i-j|} + {P_{\rm M}}^{|i-j|}\right]_{ij}$$

is positive definite, where $0^0 \triangleq 1$ and $P_{\rm M} < 1$. The result in this appendix is used in Theorem 2 to prove the strict convexity of function $\overline{v}(\cdot)$.

When $P_{\rm M} = 1$, the absorption time in Theorem 1 becomes infinite, and the receiver will never find an in-phase cell. When $P_{\rm M} = 0$, the matrix $\boldsymbol{H}$ is clearly positive definite since

$$\boldsymbol{H} = \frac{\tau_{\rm P}}{N_{\rm unc}} \boldsymbol{I}$$

where $\boldsymbol{I}$ is the identity matrix. Therefore, we will consider the case when $0 < P_{\rm M} < 1$.

We rewrite $\boldsymbol{H}$ as

$$\boldsymbol{H} = \frac{{P_{\rm M}}^{N_{\rm hit}/2} \tau_{\rm P}}{N_{\rm unc}\left(1 - {P_{\rm M}}^{N_{\rm hit}}\right)} \tilde{\boldsymbol{H}}$$

where

$$\tilde{\boldsymbol{H}} \triangleq \left[{P_{\rm M}}^{N_{\rm hit}/2-|i-j|} + {P_{\rm M}}^{-N_{\rm hit}/2+|i-j|}\right]_{ij}. \tag{45}$$

Note that the coefficient of $\tilde{\boldsymbol{H}}$ is positive. Therefore, it is sufficient to show that $\tilde{\boldsymbol{H}}$ is positive definite.

Lemma 2 (Positive-Definite Matrix): For any $P_{\rm M} \in (0, 1)$, matrix $\tilde{\boldsymbol{H}}$ defined in (45) is positive definite.

Proof: From the definition of matrix $\tilde{\boldsymbol{H}}$, it is easy to verify that $\tilde{\boldsymbol{H}}$ is circulant, symmetric, and Toeplitz for $P_{\rm M} \in (0, 1)$. Therefore, the $N_{\rm hit} \times N_{\rm hit}$ Fourier matrix

$$\boldsymbol{F} \triangleq \left[\omega^{(j-1)(k-1)}\right]_{jk} \tag{46}$$

with $\omega \triangleq {\rm e}^{2\pi\sqrt{-1}/N_{\rm hit}}$ diagonalizes $\tilde{\boldsymbol{H}}$ [66, p. 268], implying that the columns of $\boldsymbol{F}$ are the eigenvectors of $\tilde{\boldsymbol{H}}$.

Note that the first element of every eigenvector is one. Therefore, the kth eigenvector is equal to the inner product of the first row of $\tilde{\boldsymbol{H}}$ and the kth column of $\boldsymbol{F}$:

$$\lambda_k = \sum_{j=1}^{N_{\rm hit}} \tilde{\boldsymbol{H}}_{1j} \boldsymbol{F}_{jk}, \qquad k = 1, 2, \ldots, N_{\rm hit}. \tag{47}$$

Substituting $\tilde{\boldsymbol{H}}_{1j}$ and $\boldsymbol{F}_{jk}$ in (47) and simplifying terms, we have

$$\begin{aligned}
\lambda_k &= \sum_{j=1}^{N_{\rm hit}} \left({P_{\rm M}}^{N_{\rm hit}/2-|1-j|} + {P_{\rm M}}^{-N_{\rm hit}/2+|1-j|}\right) \omega^{(j-1)(k-1)} \\
&= {P_{\rm M}}^{N_{\rm hit}/2} \cdot \sum_{i=0}^{N_{\rm hit}-1} \left({P_{\rm M}}^{-1} \omega^{k-1}\right)^i \\
&\quad + {P_{\rm M}}^{-N_{\rm hit}/2} \cdot \sum_{i=0}^{N_{\rm hit}-1} \left(P_{\rm M} \omega^{k-1}\right)^i \\
&\overset{(a)}{=} {P_{\rm M}}^{N_{\rm hit}/2} \left[\frac{1 - {P_{\rm M}}^{-N_{\rm hit}}}{1 - {P_{\rm M}}^{-1}\omega^{k-1}}\right] + {P_{\rm M}}^{-N_{\rm hit}/2} \left[\frac{1 - {P_{\rm M}}^{N_{\rm hit}}}{1 - P_{\rm M}\omega^{k-1}}\right] \\
&= \frac{-\omega^{k-1}}{-{P_{\rm M}}^{-1}\omega^{k-1}\left(1 - P_{\rm M}\omega^{-(k-1)}\right)\left(1 - P_{\rm M}\omega^{k-1}\right)} \\
&\quad \times \Big({P_{\rm M}}^{-(N_{\rm hit}+2)/2} - {P_{\rm M}}^{-(N_{\rm hit}-2)/2} \\
&\qquad + {P_{\rm M}}^{(N_{\rm hit}+2)/2} - {P_{\rm M}}^{(N_{\rm hit}-2)/2}\Big) \\
&\overset{(b)}{=} \frac{\left(1 - {P_{\rm M}}^{N_{\rm hit}}\right)(1 - {P_{\rm M}}^2)}{{P_{\rm M}}^{N_{\rm hit}/2}\left|1 - P_{\rm M}\omega^{k-1}\right|^2}.
\end{aligned} \tag{48}$$

The equality (a) follows from the geometric sum formula and the fact that $\omega^{N_{\rm hit}} = 1$. The equality (b) follows from the fact that the denominator contains the product of a complex conjugate pair.

It is clear from (48) that for $P_{\rm M} \in (0, 1)$

$$\lambda_k > 0, \quad k = 1, 2, \ldots, N_{\rm hit}.$$

Since every eigenvalue of $\tilde{\boldsymbol{H}}$ is positive, the matrix $\tilde{\boldsymbol{H}}$ is positive definite [67, Theorem 7.2.1, p. 402]. That completes the proof. □

Appendix III
Solution to the Minimization Problem

The proof of Theorem 3 requires the fact that solution $\boldsymbol{x}^*$ to the relaxation problem has equal components. The precise statement of this fact is given in the following lemma.

Lemma 3 (Optimal Solution to Relaxation Problem):

1) The unique solution $\boldsymbol{x}^*$ to the optimization problem $\min_{\boldsymbol{x} \in \mathcal{S}_{\rm c}} \overline{v}(\boldsymbol{x})$ is

$$x_1^* = x_2^* = x_3^* = \cdots = x_{N_{\rm hit}}^* = \frac{N_{\rm unc}}{N_{\rm hit}} - 1. \tag{49}$$

2) The optimal cost $\min_{\boldsymbol{m} \in \mathcal{S}_{\rm c}} \overline{v}(\boldsymbol{m})$ satisfies

$$\min_{\boldsymbol{x} \in \mathcal{S}_{\rm c}} \overline{v}(\boldsymbol{x}) = \overline{v}\left(\frac{N_{\rm unc}}{N_{\rm hit}} - 1, \frac{N_{\rm unc}}{N_{\rm hit}} - 1, \ldots, \frac{N_{\rm unc}}{N_{\rm hit}} - 1\right) \tag{50}$$

$$= \left(\frac{N_{\rm unc}}{N_{\rm hit}} - 1\right)\left(\frac{1 + P_{\rm M}}{1 - P_{\rm M}}\right)\frac{\tau_{\rm P}}{2} + \frac{P_{\rm M}}{1 - P_{\rm M}}\tau_{\rm M} + \tau_{\rm D}. \tag{51}$$

3) If $N_{\rm unc}/N_{\rm hit}$ is an integer, $\boldsymbol{x}^*$ is also the unique solution to the integer programming problem $\min_{\boldsymbol{m}\in\mathcal{S}_{\rm d}} v(\boldsymbol{m})$.

Proof:

1) By Weierstrass' theorem [67, p. 541], there exists $\boldsymbol{x}^* \in \mathcal{S}_{\rm c}$ such that $\overline{v}(\boldsymbol{x}^*) \le \overline{v}(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathcal{S}_{\rm c}$. By strict convexity of $\overline{v}(\cdot)$, $\boldsymbol{x}^*$ is the unique optimal solution to the relaxation problem. Furthermore, by rotational invariance (property 2 of Theorem 2), any $\boldsymbol{x} \in \mathcal{S}_{\rm c}$ satisfies

$$\begin{aligned}\overline{v}(x_1, x_2, \ldots, x_{N_{\rm hit}}) &= \overline{v}(x_2, x_3, \ldots, x_{N_{\rm hit}}, x_1)\\ &= \overline{v}(x_3, x_4, \ldots, x_{N_{\rm hit}}, x_1, x_2)\\ &\vdots\\ &= \overline{v}(x_{N_{\rm hit}}, x_1, x_2, \ldots, x_{N_{\rm hit}-1}).\end{aligned}$$

Applying the above property to the unique solution $\boldsymbol{x}^*$, we have $x_1^* = x_2^* = x_3^* = \cdots = x_{N_{\rm hit}}^*$. Since the sum of its components is $(N_{\rm unc} - N_{\rm hit})$, the optimal solution satisfies

$$x_1^* = x_2^* = x_3^* = \cdots = x_{N_{\rm hit}}^* = N_{\rm unc}/N_{\rm hit} - 1.$$

2) Equation (50) follows immediately from part one of this lemma. Equation (51) follows from the explicit expression of $v(\cdot)$ in Theorem 1.

3) Since $\mathcal{S}_{\rm d} \subset \mathcal{S}_{\rm c}$, we have the relationship

$$\min_{\boldsymbol{m}\in\mathcal{S}_{\rm d}} v(\boldsymbol{m}) \ge \min_{\boldsymbol{x}\in\mathcal{S}_{\rm c}} \overline{v}(\boldsymbol{x}) = \overline{v}(\boldsymbol{x}^*).$$

If $N_{\rm unc}/N_{\rm hit}$ is an integer, then $\boldsymbol{x}^* \in \mathcal{S}_{\rm d}$, and the above inequality is satisfied with equality. Therefore, $\boldsymbol{x}^*$ is also the unique solution to the integer programming problem.

That completes the proof. □

Appendix IV
Anti-Symmetric Property

The proof of Theorem 3 requires the fact that the receiver cannot always distribute evenly the non-in-phase cells in the search sequence. Thus, the MAT $T_{\min}^{\rm L}$ is not always possible to achieve.

Lemma 4 (Anti-Symmetric Property): If $2 \le N_{\rm hit} \le N_{\rm unc} - 1$, then for every $\pi \in \mathcal{P}$, there is $b_0 \in \mathcal{U}$ such that

$$\boldsymbol{s}(\pi, b_0) \neq \left(\frac{N_{\rm unc}}{N_{\rm hit}} - 1, \frac{N_{\rm unc}}{N_{\rm hit}} - 1, \ldots, \frac{N_{\rm unc}}{N_{\rm hit}} - 1\right). \quad (52)$$

Proof: We will show that

$$\boldsymbol{s}(\pi, 1) \neq \left(\frac{N_{\rm unc}}{N_{\rm hit}} - 1, \frac{N_{\rm unc}}{N_{\rm hit}} - 1, \ldots, \frac{N_{\rm unc}}{N_{\rm hit}} - 1\right) \text{ or}$$

$$\boldsymbol{s}(\pi, 2) \neq \left(\frac{N_{\rm unc}}{N_{\rm hit}} - 1, \frac{N_{\rm unc}}{N_{\rm hit}} - 1, \ldots, \frac{N_{\rm unc}}{N_{\rm hit}} - 1\right).$$

Assume to the contrary that

$$\boldsymbol{s}(\pi, 1) = \left(\frac{N_{\rm unc}}{N_{\rm hit}} - 1, \frac{N_{\rm unc}}{N_{\rm hit}} - 1, \ldots, \frac{N_{\rm unc}}{N_{\rm hit}} - 1\right) \quad (53)$$

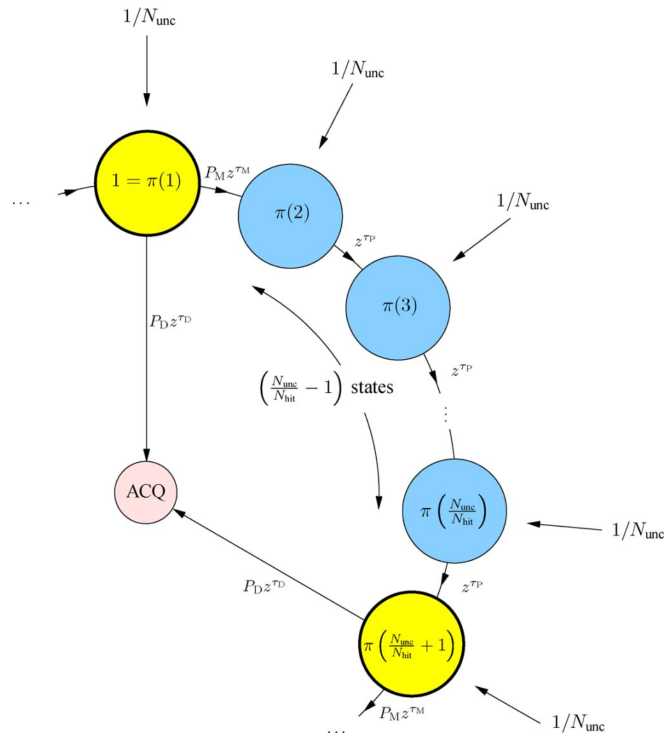

Fig. 10. The set of in-phase cells is $\mathcal{H}_{\rm hit}(1) = \{1, 2, \ldots, N_{\rm hit}\}$, and there are $(\frac{N_{\rm unc}}{N_{\rm hit}} - 1)$ H_0-states between two neighboring in-phase cells.

and

$$\boldsymbol{s}(\pi, 2) = \left(\frac{N_{\rm unc}}{N_{\rm hit}} - 1, \frac{N_{\rm unc}}{N_{\rm hit}} - 1, \ldots, \frac{N_{\rm unc}}{N_{\rm hit}} - 1\right). \quad (54)$$

Equation (53) implies that elements of

$$\mathcal{H}_{\rm hit}(1) = \{1, 2, 3, \ldots, N_{\rm hit}\} \quad (55)$$

are equally spaced in the flow diagram (see Fig. 10). Similarly, (54) implies that elements of

$$\mathcal{H}_{\rm hit}(2) = \{2, 3, 4, \ldots, N_{\rm hit}, N_{\rm hit} \oplus 1\} \quad (56)$$

are equally spaced in the flow diagram. Because $2 \le N_{\rm hit}$, we have $2 \in \mathcal{H}_{\rm hit}(1) \cap \mathcal{H}_{\rm hit}(2)$. Then, (53) implies that the $N_{\rm hit}$ elements of $\mathcal{H}_{\rm hit}(1)$ are as follows:

$$\pi\left(2 \oplus i\frac{N_{\rm unc}}{N_{\rm hit}}\right), \qquad i = 0, 1, 2, \ldots, N_{\rm hit} - 1.$$

Similarly, (54) implies that the $N_{\rm hit}$ elements of $\mathcal{H}_{\rm hit}(2)$ are as follows:

$$\pi\left(2 \oplus i\frac{N_{\rm unc}}{N_{\rm hit}}\right), \qquad i = 0, 1, 2, \ldots, N_{\rm hit} - 1.$$

Therefore, $\mathcal{H}_{\rm hit}(1) = \mathcal{H}_{\rm hit}(2)$.

Comparing elements of the two sets in (55) and (56), we have

$$1 \oplus N_{\rm hit} = 1. \quad (57)$$

Equation (57) implies that $N_{\rm hit}$ is divisible by $N_{\rm unc}$, or equivalently, $N_{\rm hit} \in \{0, N_{\rm unc}\}$. This is a contradiction since $2 \le N_{\rm hit} \le N_{\rm unc} - 1$ by the hypothesis of the lemma. That completes the proof. □

APPENDIX V
RELATIONSHIP BETWEEN η-OPTIMAL SEARCH ORDERS AND η-OPTIMAL SPACING RULES

This appendix investigates an approach for proving that a given search order is η-optimal. The result in this appendix is used to justify the proof statement of Theorem 4.

Lemma 5 (η-Isometry Property): If the search order π satisfies

$$\boldsymbol{s}(\pi^{N_{\text{hit}}}, b) \in \mathcal{S}_\eta, \qquad \text{for all } b = 1, 2, \ldots, N_{\text{hit}}.$$

for some η-optimal subset $\mathcal{S}_\eta \subset \mathcal{S}_{\text{d}}$, then π is η-optimal.

Proof: Let π be a search order that satisfies

$$\boldsymbol{s}(\pi^{N_{\text{hit}}}, b) \in \mathcal{S}_\eta, \qquad \text{for all } b = 1, 2, \ldots, N_{\text{hit}}$$

which implies that

$$\frac{v(\boldsymbol{s}(\pi, b)) - T_{\min}^{\text{L}}}{T_{\min}^{\text{L}}} \leq \eta(N_{\text{hit}}, N_{\text{unc}}) \tag{58}$$

and $\eta(N_{\text{hit}}, N_{\text{unc}}) \to 0$ as $N_{\text{hit}}/N_{\text{unc}} \to 0$. Then

$$\frac{\mathbb{E}\{\mathsf{T}_{\text{ACQ}}(\pi)\} - T_{\min}^{\text{L}}}{T_{\min}^{\text{L}}} = \frac{1}{N_{\text{unc}}} \sum_{b=1}^{N_{\text{unc}}} \left[\frac{v(\boldsymbol{s}(\pi, b)) - T_{\min}^{\text{L}}}{T_{\min}^{\text{L}}}\right] \overset{(a)}{\leq} \eta(N_{\text{unc}}, N_{\text{hit}}),$$

where (a) follows from (58). Therefore, the search order π is η-optimal. That completes the proof. □

APPENDIX VI
η-OPTIMAL SPACING RULES

A spacing rule $\boldsymbol{m} \in \mathcal{R}$ has components that are "almost equal to one another," where $\mathcal{R}$ is defined in (37). In this appendix, we show that $\mathcal{R}$ is η-optimal. The result in this appendix is used to prove Theorem 4.

Lemma 6 (η-Optimal Spacing Rules): For every $\boldsymbol{m}_{\text{r}} \in \mathcal{R}$,

$$\frac{v(\boldsymbol{m}_{\text{r}}) - T_{\min}^{\text{L}}}{T_{\min}^{\text{L}}} < \frac{2N_{\text{hit}}}{N_{\text{unc}} - N_{\text{hit}}}.$$

Thus, $\mathcal{R}$ is η-optimal with $\eta = \left(\frac{2N_{\text{hit}}}{N_{\text{unc}} - N_{\text{hit}}}\right)$.

Proof: Let

$$\boldsymbol{x}^* \triangleq \left(\frac{N_{\text{unc}}}{N_{\text{hit}}} - 1, \ldots, \frac{N_{\text{unc}}}{N_{\text{hit}}} - 1\right)$$

denote the solution of the relaxation problem in (49). That is, $T_{\min}^{\text{L}} = \overline{v}(\boldsymbol{x}^*)$. For any spacing rule $\boldsymbol{m}$, we have

$$\begin{aligned}\frac{v(\boldsymbol{m}) - T_{\min}^{\text{L}}}{T_{\min}^{\text{L}}} &= \frac{v(\boldsymbol{m}) - \overline{v}\left(\frac{N_{\text{unc}}}{N_{\text{hit}}} - 1, \ldots, \frac{N_{\text{unc}}}{N_{\text{hit}}} - 1\right)}{T_{\min}^{\text{L}}} \\ &= \frac{1}{T_{\min}^{\text{L}}}\left\{A \sum_{i=1}^{N_{\text{hit}}}\left[m_i^2 - \left(\frac{N_{\text{unc}}}{N_{\text{hit}}} - 1\right)^2\right]\right. \\ &\quad \left. + \sum_{i=1}^{N_{\text{hit}}} \sum_{j=i+1}^{N_{\text{hit}}} B_{ij}\left[m_i m_j - \left(\frac{N_{\text{unc}}}{N_{\text{hit}}} - 1\right)^2\right]\right\}. \end{aligned} \tag{59}$$

The last equality follows from (19a). For a spacing rule $\boldsymbol{m}_{\text{r}} \in \mathcal{R}$, (37) and (59) imply that

$$\begin{aligned}\frac{v(\boldsymbol{m}_{\text{r}}) - T_{\min}^{\text{L}}}{T_{\min}^{\text{L}}} &\leq \frac{1}{T_{\min}^{\text{L}}}\left\{A \sum_{i=1}^{N_{\text{hit}}}\left[\left(\frac{N_{\text{unc}}}{N_{\text{hit}}}\right)^2 - \left(\frac{N_{\text{unc}}}{N_{\text{hit}}} - 1\right)^2\right]\right. \\ &\quad \left. + \sum_{i=1}^{N_{\text{hit}}} \sum_{j=i+1}^{N_{\text{hit}}} B_{ij}\left[\left(\frac{N_{\text{unc}}}{N_{\text{hit}}}\right)^2 - \left(\frac{N_{\text{unc}}}{N_{\text{hit}}} - 1\right)^2\right]\right\} \\ &= \frac{1}{T_{\min}^{\text{L}}} \cdot \frac{2N_{\text{unc}} - N_{\text{hit}}}{N_{\text{hit}}}\left(N_{\text{hit}} A + \sum_{i=1}^{N_{\text{hit}}} \sum_{j=i+1}^{N_{\text{hit}}} B_{ij}\right) \\ &= \frac{1}{T_{\min}^{\text{L}}} \cdot \left(\frac{2N_{\text{unc}} - N_{\text{hit}}}{N_{\text{hit}}}\right) \cdot \frac{N_{\text{hit}}}{N_{\text{unc}}} \\ &\quad \times \left(\frac{1 + P_{\text{M}}}{1 - P_{\text{M}}}\right) \cdot \frac{\tau_{\text{P}}}{2} \\ &\overset{(a)}{<} \frac{\left(\frac{2N_{\text{unc}} - N_{\text{hit}}}{N_{\text{unc}}}\right) \cdot \left(\frac{1 + P_{\text{M}}}{1 - P_{\text{M}}}\right) \cdot \frac{\tau_{\text{P}}}{2}}{\left(\frac{N_{\text{unc}} - N_{\text{hit}}}{N_{\text{hit}}}\right) \cdot \left(\frac{1 + P_{\text{M}}}{1 - P_{\text{M}}}\right) \cdot \frac{\tau_{\text{P}}}{2}} \\ &< \frac{2N_{\text{hit}}}{N_{\text{unc}} - N_{\text{hit}}}. \end{aligned} \tag{60}$$

The inequality (a) follows from the explicit expression for $T_{\min}^{\text{L}}$ in (28). We note that

$$\frac{2N_{\text{hit}}}{N_{\text{unc}} - N_{\text{hit}}} \to 0$$

as $N_{\text{hit}}/N_{\text{unc}} \to 0$. Thus, the set $\mathcal{R}$ is η-optimal with $\eta = \left(\frac{2N_{\text{hit}}}{N_{\text{unc}} - N_{\text{hit}}}\right)$. That completes the proof. □

APPENDIX VII
THE SEARCH ORDER $\pi^{N_{\text{hit}}}$ AND THE CORRESPONDING SPACING RULES

The result in this appendix is used to prove Theorem 4. The goal here is to show that for every $b \in \mathcal{U}$, the description $(\pi^{N_{\text{hit}}}, b)$ maps to the spacing rule $\boldsymbol{s}(\pi^{N_{\text{hit}}}, b) \in \mathcal{R}$, where $\mathcal{R}$ is defined in (37).

Lemma 7 (Spacing Rules of $\pi^{N_{\text{hit}}}$): If N_{hit} and N_{unc} are relatively prime, then

$$\boldsymbol{s}(\pi^{N_{\text{hit}}}, b) \in \mathcal{R}, \qquad \text{for all } b = 1, 2, \ldots, N_{\text{hit}}. \tag{61}$$

Proof: Let any $b \in \mathcal{U}$ be given, and let $a \in \mathcal{H}_{\text{hit}}(b)$ be any in-phase cell. A receiver that employs the search order $\pi^{N_{\text{hit}}}$ tests the cells in the order

$$\begin{aligned}&\ldots, a,\ a \oplus N_{\text{hit}},\ a \oplus 2N_{\text{hit}}, \ldots, \\ &\qquad a \oplus j_a N_{\text{hit}},\ a \oplus (j_a + 1)N_{\text{hit}},\ \ldots \end{aligned} \tag{62}$$

where $j_a \geq 0$ in (62) denote the smallest integer such that $a \oplus (j_a + 1)N_{\text{hit}}$ is an in-phase cell.[12] Thus, in the flow diagram corresponding to $\pi^{N_{\text{hit}}}$, the number of non-in-phase cells between two neighboring in-phase cells a and $a \oplus (j_a + 1)N_{\text{hit}}$ is equal to j_a. Since a is arbitrary, it is sufficient to prove the lemma by showing that $j_a \leq \lfloor N_{\text{unc}}/N_{\text{hit}} \rfloor$.

Consider the periodic sequence (with the period N_{unc}) of consecutive cells in the uncertainty index set as shown in Fig. 11(a), where

$$S_{\text{A}} \triangleq b,\ b \oplus 1,\ b \oplus 2,\ \ldots,\ a,\ \ldots,\ b \oplus (N_{\text{hit}} - 1) \tag{63}$$

[12]Note that j_a is written explicitly with the subscript a to indicate its dependence on the specific in-phase cell a.

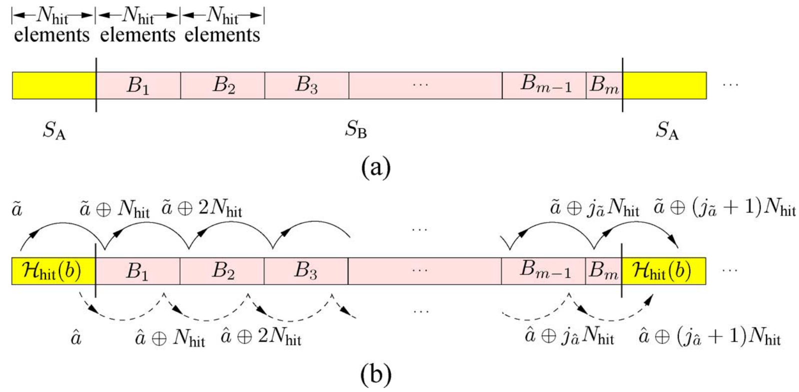

Fig. 11. Sequences S_A and S_B are sequences of cells in the uncertainty index sets. (a) The set of non-in-phase cells is partitioned into m subsets: $B_1, B_2, \ldots, B_m$. (b) The number ($j_{\tilde{a}}$ or $j_{\hat{a}}$) of non-in-phase cells between two neighboring in-phase cells cannot exceed the number m of subsets: $j_{\tilde{a}} \le m$ and $j_{\hat{a}} \le m$.

and

$$S_{\text{B}} \triangleq b \oplus N_{\text{hit}},\, b \oplus (N_{\text{hit}}+1),\, \ldots,\, b \oplus (-1). \tag{64}$$

The sequences S_A and S_B correspond to the sequence of in-phase cells and the sequence of non-in-phase cells, respectively. We partition S_B into nonempty $m \ge 1$ subsets $B_1, B_2, \ldots, B_m$, with

$$|B_1| = |B_2| = \cdots = |B_{m-1}| = N_{\text{hit}} \quad \text{and} \quad 1 \le |B_m| \le N_{\text{hit}}. \tag{65}$$

Because there are $N_{\text{unc}} - N_{\text{hit}}$ non-in-phase cells, the number m of subsets is

$$m = \lceil (N_{\text{unc}} - N_{\text{hit}})/N_{\text{hit}} \rceil \tag{66}$$
$$= \lfloor N_{\text{unc}}/N_{\text{hit}} \rfloor. \tag{67}$$

The last equality follows from the fact that N_{unc} and N_{hit} are relatively prime.

There are two possible cases for the cell $a \oplus N_{\text{hit}}$. In the first case, the cell $a \oplus N_{\text{hit}}$ is an in-phase cell. Then, $j_a = 0$ and the inequality $j_a \le \lfloor N_{\text{unc}}/N_{\text{hit}} \rfloor$ is immediately satisfied. In the second case, the cell $a \oplus N_{\text{hit}}$ is a non-in-phase cell. It is not hard to verify that

$$a \oplus N_{\text{hit}} \in B_1,\, a \oplus 2N_{\text{hit}} \in B_2,\, a \oplus 3N_{\text{hit}} \in B_3,\, \ldots,\, a \oplus j_a N_{\text{hit}} \in B_l,\, a \oplus (j_a+1)N_{\text{hit}} \in \mathcal{H}_{\text{hit}}(b) \tag{68}$$

where l is either m or $m-1$ (see Fig. 11(b)). In other words, when the receiver advances its cell by N_{hit} steps, it "moves" from the current subset B_i, for some i, to the adjacent subset B_{i+1} or to the set $\mathcal{H}_{\text{hit}}(b)$. Equation (68) and the fact that $l \in \{m-1, m\}$ imply that $j_a = l \le m$. Substituting the expression for m in (67), we have the inequality $j_a \le \lfloor N_{\text{unc}}/N_{\text{hit}} \rfloor$. That completes the proof. □

Appendix VIII
Solution to the Maximization Problem

The proof of Theorem 5 requires the fact that consecutive non-in-phase cells in the flow diagram results in the maximum absorption time. The precise statement of this fact is given in the next lemma.

Lemma 8 (Maximum Absorption Time):

1) The complete solutions to the integer programming problem $\max_{\boldsymbol{m} \in \mathcal{S}_d} v(\boldsymbol{m})$ are elements of $\mathcal{E}$, where $\mathcal{E}$ is defined in (15).
2) The maximum absorption time is equal to

$$\max_{\boldsymbol{m} \in \mathcal{S}_d} v(\boldsymbol{m}) = \frac{(N_{\text{unc}} - N_{\text{hit}})^2}{N_{\text{unc}}} \cdot \left(\frac{1 + P_M{}^{N_{\text{hit}}}}{1 - P_M{}^{N_{\text{hit}}}} \right) \frac{\tau_P}{2} + \left(1 - \frac{N_{\text{hit}}}{N_{\text{unc}}}\right) \cdot \left(\frac{1 + P_M}{1 - P_M} \right) \frac{\tau_P}{2} + \frac{P_M}{1 - P_M} \tau_M + \tau_D. \tag{69}$$

Proof:

1) Let any spacing rule $\boldsymbol{m} \triangleq (m_1, m_2, \ldots, m_{N_{\text{hit}}}) \notin \mathcal{E}$ be given. Note that $\mathcal{E}$ is a set of extreme points of the bounded polyhedron $\mathcal{S}_c$ (see Lemma 1 in Appendix I). By the Resolution Theorem [64, p. 179], the spacing rule $\boldsymbol{m} \in \mathcal{S}_d \subset \mathcal{S}_c$ can be written as a convex combination of the extreme points of $\mathcal{S}_c$

$$\boldsymbol{m} = \sum_{i=1}^{N_{\text{hit}}} \lambda_i \boldsymbol{m}^{(i)}$$

where $\lambda_i \ge 0$, $\sum_{i=1}^{N_{\text{hit}}} \lambda_i = 1$, and $\boldsymbol{m}^{(i)}$ is defined in (16). Then

$$\overline{v}(\boldsymbol{m}) \overset{(a)}{<} \sum_{i=1}^{N_{\text{hit}}} \lambda_i \overline{v}(\boldsymbol{m}^{(i)}) \overset{(b)}{=} \overline{v}(\boldsymbol{m}^{(1)}).$$

The inequality (a) follows from strict convexity of $\overline{v}(\cdot)$. The equality (b) follows from rotational invariance of $\overline{v}(\cdot)$ (the second property of Theorem 2): $\overline{v}(\boldsymbol{m}^{(1)}) = \cdots = \overline{v}(\boldsymbol{m}^{(N_{\text{hit}})})$. Notice that any spacing rule $\boldsymbol{m}_0 \in \mathcal{S}_d$ satisfies $\overline{v}(\boldsymbol{m}_0) = v(\boldsymbol{m}_0)$. Thus, for any $\boldsymbol{m} \notin \mathcal{E}$

$$v(\boldsymbol{m}) < v(\boldsymbol{m}^{(1)}) = \overline{v}(\boldsymbol{m}^{(2)}) = \cdots = \overline{v}(\boldsymbol{m}^{(N_{\text{hit}})}). \tag{70}$$

Therefore, $\mathcal{E}$ contains all solutions to the maximization problem.

2) This part of the lemma follows immediately from part one and the explicit expression of $v(\cdot)$ in Theorem 1.

That completes the proof. □

APPENDIX IX
PROPERTIES OF SEARCH ORDERS THAT ACHIEVE THE MAXIMUM MAT

In this appendix, we prove two important properties of a search order π_{w} that exhibits the maximum MAT. These properties are used in Theorem 5 to find search orders that result in the maximum MAT.

Definition 6 (π-Cluster): A set of cells is called a *π-cluster* if its elements are adjacent to each other in the flow diagram corresponding to π.

Lemma 9 (Weak Clustering Property): Let any N_{unc} and N_{hit} be given. Let π_{w} denote any search order that maximizes the MAT. Then, for every $b_0 \in \mathcal{U}$, there exists some $k \in \mathcal{U}$ such that

$$\{b_0 \oplus i \,|\, 0 \le i < N_{\mathrm{hit}}\} = \{\pi_{\mathrm{w}}(k \oplus i) \,|\, 0 \le i < N_{\mathrm{hit}}\}.$$

Proof: Let N_{unc}, N_{hit}, and π_{w} that satisfy the lemma statement be given. Let $b_0 \in \mathcal{U}$ be given. Then

$$\begin{aligned} T_{\max} &= \mathbb{E}\{\mathsf{T}_{\mathrm{ACQ}}(\pi_{\mathrm{w}})\} \\ &= \frac{1}{N_{\mathrm{unc}}} \sum_{b=1}^{N_{\mathrm{unc}}} v(\boldsymbol{s}(\pi_{\mathrm{w}}, b)) \\ &\overset{(a)}{\le} \frac{1}{N_{\mathrm{unc}}} \sum_{b=1}^{N_{\mathrm{unc}}} \max_{\boldsymbol{m} \in \mathcal{S}_{\mathrm{d}}} v(\boldsymbol{m}) \\ &= \max_{\boldsymbol{m} \in \mathcal{S}_{\mathrm{d}}} v(\boldsymbol{m}) \\ &= T_{\max}. \end{aligned} \tag{71}$$

The last equality follows from part two of Lemma 8 and (38). Therefore, the inequality (a) is satisfied with equality

$$\frac{1}{N_{\mathrm{unc}}} \sum_{b=1}^{N_{\mathrm{unc}}} v(\boldsymbol{s}(\pi_{\mathrm{w}}, b)) = \max_{\boldsymbol{m} \in \mathcal{S}_{\mathrm{d}}} v(\boldsymbol{m}).$$

In other words, the average of N_{unc} absorption times is equal to the maximum absorption time. Thus, every absorption time must equal the maximum one, and, in particular

$$v(\boldsymbol{s}(\pi_{\mathrm{w}}, b_0)) = \max_{\boldsymbol{m} \in \mathcal{S}_{\mathrm{d}}} v(\boldsymbol{m}). \tag{72}$$

By part one of Lemma 8, (72) implies that $\boldsymbol{s}(\pi_{\mathrm{w}}, b_0) \in \mathcal{E}$. Recall that every spacing rule in $\mathcal{E}$ corresponds to a flow diagram with consecutive in-phase cells. Therefore, there exits $k \in \mathcal{U}$ such that

$$\mathcal{H}_{\mathrm{hit}}(b_0) = \{\pi_{\mathrm{w}}(k \oplus i) \,|\, 0 \le i < N_{\mathrm{hit}}\}.$$

That completes the proof. □

Lemma 10 (Strong Clustering Property): Let N_{hit} and N_{unc} satisfy $2 \le N_{\mathrm{hit}} \le N_{\mathrm{unc}} - 2$. Let π_{w} denote any search order that maximizes the MAT. Then, for every *cluster size* N_{clu} satisfying $1 \le N_{\mathrm{clu}} \le N_{\mathrm{hit}}$ and for every $b_0 \in \mathcal{U}$, there exists some $k \in \mathcal{U}$ such that

$$\{b_0 \oplus i \,|\, 0 \le i < N_{\mathrm{clu}}\} = \{\pi_{\mathrm{w}}(k \oplus i) \,|\, 0 \le i < N_{\mathrm{clu}}\}.$$

Proof: Let N_{unc}, N_{hit}, π_{w}, b_0, and k satisfy statement in the lemma. We will show that

$$\begin{aligned} \forall (b \in \mathcal{U})\, \exists (k \in \mathcal{U}) \ \ni \{b \oplus i \,|\, 0 \le i < N_{\mathrm{clu}}\} \\ = \{\pi_{\mathrm{w}}(k \oplus i) \,|\, 0 \le i < N_{\mathrm{clu}}\} \end{aligned} \tag{73}$$

by induction on N_{clu}.

- Base case ($N_{\mathrm{clu}} = N_{\mathrm{hit}}$).
  The condition (73) is satisfied by Weak Clustering Property (Lemma 9).
- Inductive step.
  Let any $b_0 \in \mathcal{U}$ be given. Assume that the condition (73) is satisfied for some $2 \le N_{\mathrm{clu}} \le N_{\mathrm{hit}}$. The inductive hypothesis implies that there exists $k \in \mathcal{U}$ such that

$$\{b_0 \oplus i \,|\, 0 \le i < N_{\mathrm{clu}}\} = \{\pi_{\mathrm{w}}(k \oplus i) \,|\, 0 \le i < N_{\mathrm{clu}}\}. \tag{74}$$

It is sufficient to show the condition (73) for $N_{\mathrm{clu}} - 1$ by proving that $b_0 \oplus (N_{\mathrm{clu}} - 1) = \pi_{\mathrm{w}}(k)$ or $b_0 \oplus (N_{\mathrm{clu}} - 1) = \pi_{\mathrm{w}}(k \oplus (N_{\mathrm{clu}} - 1))$.

Assume to the contrary that $b_0 \oplus (N_{\mathrm{clu}} - 1) \ne \pi_{\mathrm{w}}(k)$ and $b_0 \oplus (N_{\mathrm{clu}} - 1) \ne \pi_{\mathrm{w}}(k \oplus (N_{\mathrm{clu}} - 1))$. Then, there exists $1 \le d \le N_{\mathrm{clu}} - 2$ such that $b_0 \oplus (N_{\mathrm{clu}} - 1) = \pi_{\mathrm{w}}(k \oplus d)$. Removing $b_0 \oplus (N_{\mathrm{clu}} - 1)$ from both sides of (74) yields

$$\begin{aligned} &\{b_0 \oplus i \,|\, 0 \le i < N_{\mathrm{clu}} - 1\} \\ &\quad = \underbrace{\{\pi_{\mathrm{w}}(k \oplus i) \,|\, 0 \le i \le d - 1\}}_{\triangleq \mathcal{A}} \\ &\qquad \bigcup \underbrace{\{\pi_{\mathrm{w}}(k \oplus i) \,|\, d + 1 \le i < N_{\mathrm{clu}}\}}_{\triangleq \mathcal{B}}. \end{aligned} \tag{75}$$

Since $1 \le d \le N_{\mathrm{clu}} - 2$, the sets $\mathcal{A}$ and $\mathcal{B}$ are nonempty. Inserting $b_0 \ominus 1$ to both sides of (75) yields

$$\{b_0 \oplus i \,|\, -1 \le i < N_{\mathrm{clu}} - 1\} = \mathcal{A} \bigcup \mathcal{B} \bigcup \{b_0 \ominus 1\}. \tag{76}$$

Note that the left-hand side of (76) is a set of N_{clu} consecutive numbers (in N_{unc}-arithmetic sense). Therefore, the inductive hypothesis implies that there exists $l \in \mathcal{U}$

$$\begin{aligned} &\{b_0 \oplus i \,|\, -1 \le i < N_{\mathrm{clu}} - 1\} \\ &\quad = \underbrace{\left\{\pi_{\mathrm{w}}(l \oplus j) \,\middle|\, 0 \le j < N_{\mathrm{clu}}\right\}}_{\triangleq \mathcal{C}}. \end{aligned} \tag{77}$$

Equating right-hand sides of (76) and (77) gives

$$\begin{aligned} &\underbrace{\left\{\pi_{\mathrm{w}}(l \oplus j) \,\middle|\, 0 \le j < N_{\mathrm{clu}}\right\}}_{\mathcal{C}} \\ &\quad = \underbrace{\{\pi_{\mathrm{w}}(k \oplus i) \,|\, 0 \le i \le d - 1\}}_{\mathcal{A}} \\ &\qquad \bigcup \underbrace{\{\pi_{\mathrm{w}}(k \oplus i) \,|\, d + 1 \le i < N_{\mathrm{clu}}\}}_{\mathcal{B}} \bigcup \{b_0 \ominus 1\}. \end{aligned} \tag{78}$$

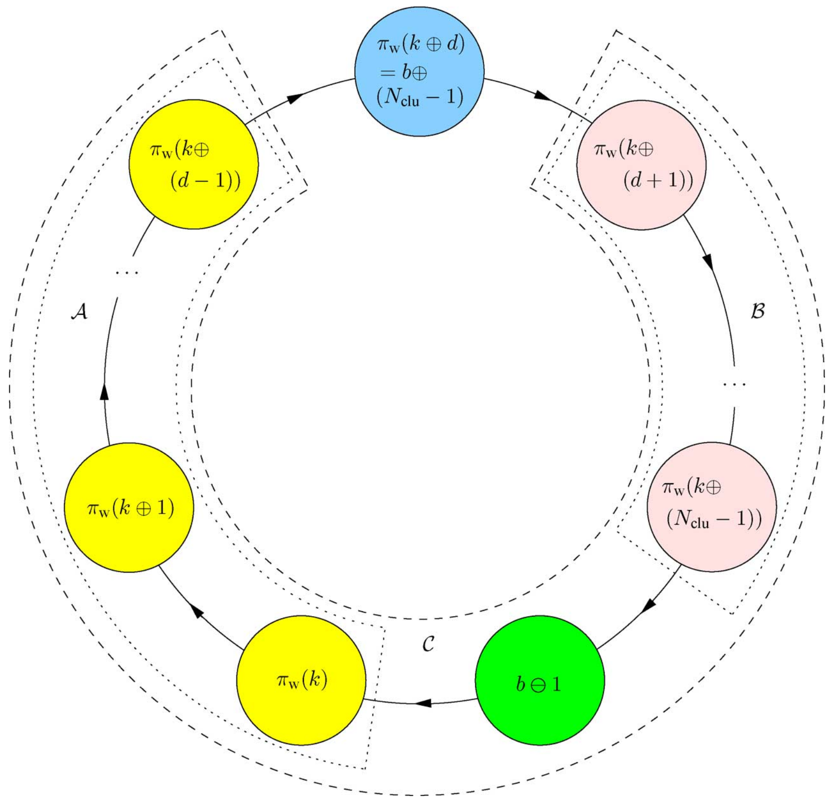

Fig. 12. Set $\mathcal{C}$ is a π_w-cluster. Therefore, the cell $b \oplus (N_{clu} - 1)$ must follow cells in $\mathcal{B}$ and precede cells in $\mathcal{A}$.

By the hypothesis of the lemma and by the range of N_{clu}, we have $N_{clu} \neq 0$ and $N_{clu} \neq N_{unc}$, which imply that $b_0 \oplus (N_{clu} - 1) \neq b_0 \ominus 1$. Substituting $b_0 \oplus (N_{clu} - 1) = \pi_w(k \oplus d)$, we have (see Fig. 12)

$$\pi_w(k \oplus d) \neq b_0 \ominus 1.$$

Note that the sets in both sides of (78) are π_w-clusters. Thus, cell $b \ominus 1$ must follow those cells in $\mathcal{B}$ and precede those cells in $\mathcal{A}$ in the search sequence

$$\begin{aligned} \pi_w(k \oplus N_{clu}) &= b \ominus 1 \quad \text{(follow)} \\ \pi_w(k \ominus 1) &= b \ominus 1 \quad \text{(precede)}. \end{aligned} \tag{79}$$

Bijectivity of $\pi_w(\cdot)$, together with (79), gives $k \oplus (N_{clu} + 1) = k$, which implies that $N_{clu} + 1$ is divisible by N_{unc}. Thus, $N_{unc} \leq N_{clu} + 1$. This is a contradiction, because the hypotheses for the ranges of N_{clu} in the inductive step and N_{hit} in the lemma imply that $N_{clu} + 1 \leq N_{hit} + 1 \leq N_{unc} - 1$. That completes the proof. □

ACKNOWLEDGMENT

The authors wish to thank L. A. Shepp, G. J. Foschini, R. G. Gallager, H. V. Poor, G. E. Corazza, W. M. Gifford, and I. Keliher for their helpful suggestions and careful reading of the manuscript.

REFERENCES

[1] R. L. Pickholtz, D. L. Schilling, and L. B. Milstein, "Theory of spread spectrum communications–A tutorial," *IEEE Trans. Commun.*, vol. COM-30, no. 5, pp. 855–884, May 1982.
[2] J. K. Holmes, *Coherent Spread Spectrum Systems*, 1st ed. New York: Wiley-Interscience, 1982.
[3] M. K. Simon, J. K. Omura, R. A. Scholtz, and B. K. Levitt, *Spread Spectrum Communications Handbook*, electronic ed. New York: McGraw-Hill, Inc., 2001.
[4] R. L. Peterson, R. E. Ziemer, and D. E. Borth, *Introduction to Spread Spectrum Communications*, 1st ed. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[5] A. J. Viterbi, *CDMA: Principles of Spread Spectrum Communication*, 1st ed. Reading, MA: Addison-Wesley, 1995.
[6] A. Polydoros and S. Glisic, "Code synchronization: A review of principles and techniques," in *Proc. IEEE Int. Symp. Spread Spectrum Techniques and Applications*, Oulu, Finland, Jul. 1994, pp. 115–137.
[7] S. Gezici, E. Fishler, H. Kobayashi, H. V. Poor, and A. F. Molisch, "A rapid acquisition technique for impulse radio," in *Proc. IEEE Pacific Rim Conf. on Communication, Computers and Signal Processing*, Victoria, BC, Canada, Aug. 2003, vol. 2, pp. 627–630.
[8] M. Z. Win and Z. A. Kostić, "Virtual path analysis of selective RAKE receiver in dense multipath channels," *IEEE Commun. Lett.*, vol. 3, no. 11, pp. 308–310, Nov. 1999.
[9] M. Z. Win, G. Chrisikos, and N. R. Sollenberger, "Performance of Rake reception in dense multipath channels: Implications of spreading bandwidth and selection diversity order," *IEEE J. Select. Areas Commun.*, vol. 18, no. 8, pp. 1516–1525, Aug. 2000.
[10] M. Z. Win and R. A. Scholtz, "On the robustness of ultra-wide bandwidth signals in dense multipath environments," *IEEE Commun. Lett.*, vol. 2, no. 2, pp. 51–53, Feb. 1998.
[11] ——, "On the energy capture of ultra-wide bandwidth signals in dense multipath environments," *IEEE Commun. Lett.*, vol. 2, no. 9, pp. 245–247, Sep. 1998.
[12] ——, "Characterization of ultra-wide bandwidth wireless indoor communications channel: A communication theoretic view," *IEEE J. Select. Areas Commun.*, vol. 20, no. 9, pp. 1613–1627, Dec. 2002.
[13] D. Cassioli, M. Z. Win, and A. F. Molisch, "The ultra-wide bandwidth indoor channel: From statistical model to simulations," *IEEE J. Select. Areas Commun.*, vol. 20, no. 6, pp. 1247–1257, Aug. 2002.
[14] C.-C. Chong and S. K. Yong, "A generic statistical-based UWB channel model for high-rise apartments," *IEEE Trans. Antennas Propagat.*, vol. 53, no. 8, pp. 2389–2399, Aug. 2005.
[15] C.-C. Chong, Y.-E. Kim, S. K. Yong, and S.-S. Lee, "Statistical characterization of the UWB propagation channel in indoor residential environment," in *Wireless Communications and Mobile Computing*. New York: Wiley Interscience, 2005, vol. 5, pp. 503–512.

[16] R. R. Rick and L. B. Milstein, "Parallel acquisition in mobile DS-CDMA systems," *IEEE Trans. Commun.*, vol. 45, no. 11, pp. 1466–1476, Nov. 1997.

[17] E. Sourour and S. C. Gupa, "Direct-sequence spread spectrum parallel acquisition in nonselective and frequency-selective fading Rician fading channels," *IEEE J. Select. Areas Commun.*, vol. 10, no. 3, pp. 535–544, Apr. 1992.

[18] A. Polydoros and C. L. Weber, "A unified approach to serial search spread-spectrum code acquisition—Part I: General theory," *IEEE Trans. Commun.*, vol. COM-32, no. 5, pp. 542–549, May 1984.

[19] ——, "A unified approach to serial search spread-spectrum code acquisition—Part II: A matched-filter receiver," *IEEE Trans. Commun.*, vol. COM-32, no. 5, pp. 550–560, May 1984.

[20] A. Polydoros and M. K. Simon, "Generalized serial search code acquisition: The equivalent circular state diagram approach," *IEEE Trans. Commun.*, vol. COM-32, no. 12, pp. 1260–1268, Dec. 1984.

[21] W.-H. Sheen and H.-C. Wang, "A new analysis of direct-sequence psudonoise code acquisition on Rayleigh fading channels," *IEEE J. Select. Areas Commun.*, vol. 19, no. 11, pp. 2225–2232, Nov. 2001.

[22] W.-H. Sheen, J.-K. Tzeng, and C.-K. Tzou, "Effects of cell correlations in a mathched-filter PN code acquisition for direct-sequence spread-spectrum system," *IEEE Trans. Veh. Technol.*, vol. 48, no. 3, pp. 724–732, May 1999.

[23] W.-H. Sheen and S. Chiou, "Performance of multiple-dwell pseudo-noise code acquisition with I-Q detector on frequency-nonselective multipath fading channels," *Wireless Netw.*, vol. 5, pp. 11–21, Sep. 1999.

[24] M. D. Katz and S. Glisic, "Modeling of code acquisition process in CDMA networks–quasisynchronization systems," *IEEE Trans. Commun.*, vol. 46, no. 12, pp. 1564–1568, Dec. 1998.

[25] ——, "Modeling of code acquisition process in CDMA networks–Asynchronous systems," *IEEE J. Select. Areas Commun.*, vol. 18, no. 1, pp. 73–86, Jan. 2000.

[26] S. Glisic and M. D. Katz, "Modeling of code acquisition process for Rake receivers in CDMA wireless networks with multipath and transmitter diversity," *IEEE J. Select. Areas Commun.*, vol. 19, no. 1, pp. 21–32, Jan. 2001.

[27] M. D. Katz, J. H. Iinatti, and S. Glisic, "Two-dimensional code acquisition in environments with a spatially nonuniform distribution of interference: Algorithm and performance," *IEEE Trans. Wireless Commun.*, vol. 3, no. 1, pp. 1–7, Jan. 2004.

[28] W. Zhuang, "Noncoherent hybrid parallel PN code acquisition for CDMA mobile communications," *IEEE Trans. Veh. Technol.*, vol. 45, no. 6, pp. 643–656, Nov. 1996.

[29] M. D. Katz, J. H. Iinatti, and S. Glisic, "Two-dimensional code acquisition in time and angular domains," *IEEE J. Select. Areas Commun.*, vol. 19, no. 12, pp. 2441–2451, Dec. 2001.

[30] ——, "Recent advances in multi-antenna based code acquisition," in *Proc. IEEE Int. Symp. Spread Spectrum Techniques and Applications*, Sydney, Australia, Aug. 2004, pp. 199–206.

[31] S. Vijayakumaran and T. F. Wong, "Best permutation search strategy for ultra-wideband signal acquisition," in *Proc. Vehicular Technology Conf.*, Los Angeles, CA, Sep. 2004, pp. 1184–1187.

[32] E. A. Homier and R. A. Scholtz, "Hybrid fixed-dwell-time search techniques for rapid acquisition of ultra-wideband signals," in *International Workshop on Ultra Wideband System*, Oulu, Finland, Jun. 2003.

[33] O.-S. Shin and K. B. E. Lee, "Utilization of multipaths for spread-spectrum code acquisition in frequency-selective Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 49, no. 4, pp. 734–743, Apr. 2001.

[34] L.-L. Yang and L. Hanzo, "Serial acquisition of DS-CDMA signals in multipath fading mobile channels," *IEEE Trans. Veh. Technol.*, vol. 50, no. 2, pp. 617–628, Mar. 2001.

[35] ——, "Serial acquisition performance of single-carrier and multicarrier DS-CDMA over Nakagami-m fading channels," *IEEE Trans. Wireless Commun.*, vol. 1, no. 4, pp. 692–702, Oct. 2002.

[36] W.-H. Sheen and C.-C. Tseng, "Code acquisition with diversity combining for FFH/MFSK spread-spectrum systems," *Proc. IEE –Commun.*, vol. 48, pp. 437–444, Dec. 1998.

[37] W.-H. Sheen, C.-C. Tseng, and H.-C. Wang, "Multiple-dwell and sequential code acquisition with diversity for FFH/MFSK spread-spectrum systems under band multitone jamming," *IEEE Trans. Commun.*, vol. 48, no. 5, pp. 841–851, May 2000.

[38] R. R. Rick and L. B. Milstein, "Optimal decision strategies for acquisition of spread-spectrum signals in frequency-selective fading channels," *IEEE Trans. Commun.*, vol. 46, no. 5, pp. 686–694, May 1998.

[39] C. Carbonelli, A. A. D'Amico, U. Mengali, and M. Morelli, "Channel acquisition and tracking for DS-CDMA uplink transmissions," *IEEE Trans. Commun.*, vol. 53, no. 11, pp. 1930–1939, Nov. 2005.

[40] C. Carbonelli and U. Mengali, "Synchronization algorithms for UWB signals," *IEEE Trans. Commun.*, vol. 54, no. 2, pp. 329–338, Feb. 2006.

[41] C. W. Baum and V. V. Veeravalli, "A sequential procedure for multihypothesis testing," *IEEE Trans. Inf. Theory*, vol. 40, no. 6, pp. 1994–2007, Nov. 1994.

[42] V. V. Veeravalli and C. W. Baum, "Asymptotic efficiency of a sequential multihypothesis test," *IEEE Trans. Inf. Theory*, vol. 41, no. 6, pp. 1994–1997, Nov. 1995.

[43] V. P. Dragalin, A. G. Tartakovsky, and V. V. Veeravalli, "Multihypothesis sequential probability ratio tests – Part I: Asymptotic optimality," *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 2448–2461, Nov. 1999.

[44] ——, "Multihypothesis sequential probability ratio tests—Part II: Accurate asymptotic expansions for the expected sample size," *IEEE Trans. Inf. Theory*, vol. 46, no. 4, pp. 1366–1383, Jul. 2000.

[45] H.-C. Wang and W.-H. Sheen, "Variable dwell-time code acquisition for direct-sequence spread-spectrum systems on time-variant Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 48, no. 6, pp. 1037–1046, Jun. 2000.

[46] G. E. Corazza, "On the MAX/TC criterion for code acquisition and its application to DS-SSMA systems," *IEEE Trans. Commun.*, vol. 44, no. 9, pp. 1173–1182, Sep. 1996.

[47] G. E. Corazza and A. Polydoros, "Code acquisition in CDMA cellular mobile networks–Part I: Theory," in *Proc. IEEE 5th Int. Symp. Spread Spectrum Techniques & Applications*, Sun City, South Africa, Sep. 1998, pp. 454–458.

[48] G. E. Corazza, C. Caini, A. Vanelli-Coralli, and A. Polydoros, "DS-CDMA code acquisition in the presence of correlated fading—Part I: Theoretical aspects," *IEEE Trans. Commun.*, vol. 52, no. 7, pp. 1160–1168, Jul. 2004.

[49] ——, "DS-CDMA code acquisition in the presence of correlated fading—Part II: Application to cellular networks," *IEEE Trans. Commun.*, vol. 52, no. 8, pp. 1397–1407, Aug. 2004.

[50] W. R. Braun, "Performance analysis for the expanding search PN acquisition algorithm," *IEEE Trans. Commun.*, vol. COM-30, no. 3, pp. 424–435, Mar. 1982.

[51] A. Weinberg, "Generalized analysis for the evaluation of search strategy effects on PN acquisition performance," *IEEE Trans. Commun.*, vol. COM-31, no. 1, pp. 37–49, Jan. 1983.

[52] W. Suwansantisuk, "Multipath Aided Rapid Acquisition," Master's thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA, 2004.

[53] Y. C. Yoon, R. Kohno, and H. Imai, "A spread-spectrum multiaccess system with cochannel interference cancellation for multipath fading channels," *IEEE J. Select. Areas Commun.*, vol. 11, no. 7, pp. 1067–1075, Sep. 1993.

[54] B. R. Vojčić and R. L. Pickholtz, "Performance of coded direct sequence spread spectrum in a fading dispersive channel with pulsed jamming," *IEEE J. Select. Areas Commun.*, vol. 8, no. 7, pp. 934–942, Sep. 1990.

[55] G. L. Turin, "The effects of multipath and fading on the performance of direct-sequence CDMA systems," *IEEE J. Select. Areas Commun.*, vol. SAC-2, no. 4, pp. 597–603, Jul. 1984.

[56] L. M. Jalloul and J. M. Holtzman, "Performance analysis of DS/CDMA with noncoherent M-ary orthogonal modulation in multipath fading channels," *IEEE J. Select. Areas Commun.*, vol. 12, no. 5, pp. 862–870, Jun. 1994.

[57] G. Turin, F. D. Clapp, T. L. Johnston, S. B. Fine, and D. Lavry, "A statistical model of urban multipath propagation," *IEEE Trans. Veh. Technol.*, vol. VT-21, no. 1, pp. 1–9, Feb. 1972.

[58] H. Hashemi, "Simulation of the urban radio propagation channel," *IEEE Trans. Veh. Technol.*, vol. VT-28, no. 3, pp. 213–225, Aug. 1979.

[59] D. C. Cox and R. P. Leck, "Distributions of multipath delay spread and average excess delay for 910-MHz urban mobile radio paths," *IEEE Trans. Antennas Propagat.*, vol. AP-23, no. 3, pp. 206–213, Mar. 1975.

[60] P. F. Driessen, "Prediction of multipath delay profiles in mountainous terrain," *IEEE J. Select. Areas Commun.*, vol. 18, no. 3, pp. 336–346, Mar. 2000.

[61] W. Suwansantisuk, M. Z. Win, and L. A. Shepp, "On the performance of wide-bandwidth signal acquisition in dense multipath channels," *IEEE Trans. Veh. Technol.*, vol. 54, no. 5, pp. 1584–1594, Sep. 2005.

[62] L. A. Shepp, 1999, AT&T Labs-Research, Red Bank, NJ, private conversation.

[63] M. S. Klamkin and D. J. Newman, "The philosophy and applications of transform theory," 1960, unpublished manuscript.

[64] D. Bertsimas and J. N. Tsitsiklis, *Introduction to Linear Optimization*, 1st ed. Belmont, MA: Athena Scientific, 1997.

[65] D. P. Bertsekas, *Nonlinear Programming*. Belmont, MA: Athena Scientific, 1995.

[66] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, 1st ed. Wellesley, MA: Wellesley-Cambridge Univ. Press, 1996.

[67] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 1st ed. Cambridge, U.K.: Cambridge Univ. Press, 1990.