Generalized Interference Alignment—Part I: Theoretical Framework

Liangzhong Ruan, *Member, IEEE*, Vincent K. N. Lau, *Fellow, IEEE*, and Moe Z. Win, *Fellow, IEEE*

***Abstract*—Interference alignment (IA) has attracted enormous research interest as it achieves optimal capacity scaling with respect to signal to noise ratio on interference networks. IA has also recently emerged as an effective tool in engineering interference for secrecy protection on wireless wiretap networks. However, despite the numerous works dedicated to IA, two of its fundamental issues, i.e., feasibility conditions and transceiver design, are not completely addressed in the literature. In this two part paper, a generalized interference alignment (GIA) technique is proposed to enhance the IA's capability in secrecy protection. A theoretical framework is established to analyze the two fundamental issues of GIA in Part I and then the performance of GIA in large-scale stochastic networks is characterized to illustrate how GIA benefits secrecy protection in Part II. The theoretical framework for GIA adopts methodologies from algebraic geometry, determines the necessary and sufficient feasibility conditions of GIA, and generates a set of algorithms for solving the GIA problem. This framework sets up a foundation for the development and implementation of GIA.**

***Index Terms*— MIMO, interference alignment, algebraic geometry.**

I. INTRODUCTION

A. Background and Survey

INTERFERENCE is a major factor that limits the performance of wireless communication networks. Conventional interference control schemes, most of which adopt the principle of channel orthogonalization are in general non-capacity achieving [1], [2]. IA reduces the effect of aggregated interference by aligning interference from multiple sources into lower-dimensional subspaces at receivers [3]. It achieves the optimal capacity scaling with respect to the signal to noise ratio (SNR) in interference networks [4]. On the other hand, in a wireless network that requires the secure exchange of confidential messages, interference, which enables legitimate partners to impede the eavesdropping receivers (ERs), emerges as a potentially valuable resource for wireless network secrecy [5], [6]. In order to impede the ERs without interfering with legitimate receivers (LRs), a few studies have adopted the IA scheme proposed in [4] to promote wireless secrecy [7], [8]. However, the scheme in [4] is based on infinite-dimensional symbol extension, making it difficult to implement in practice.

To avoid the infinite-dimension issue, researchers have developed spatial-domain IA schemes, in which interference is coordinated and canceled via the finite signal dimension provided by multiple antennas. For this scheme, there are two fundamental issues: (1) When is IA (without symbol extension) feasible; and (2) Given that IA is feasible, how to design an algorithm to find transceivers (precoders and decoders) that cancel all interference? For the feasibility issue, the pioneering works characterize the IA feasibility conditions under some special configurations [9]–[12]. In [13], a numerical test that checks IA feasibility is proposed. In the authors' prior work [14], a sufficient IA feasibility condition is proved for MIMO interference networks with a general configuration. This results unifies and extends those in [9]–[11]. For the transceiver design issue, there are two categories of algorithms: constructive and iterative. The constructive algorithms apply to networks with special configurations [15]–[17]. The iterative algorithms [18]–[20] apply to networks with general configurations, but they converge to local optimums. Table I and II in Section II summarize the contributions and limitations of the existing works on IA.

Furthermore, as will be discussed in detail in Part II, to promote the capability of IA in secrecy protection, it is desirable to introduce legitimate jammers (LJs) into the network and jointly coordinate the transmission policy of all legitimate partners to create stronger interference at the ERs without affecting the LRs. In this paper, this technique is referred to as GIA. To develop such a technique, the following challenges need to be addressed:

- **Determine the feasibility conditions of GIA**: Feasibility analysis of IA is challenging because IA constraints are sets of nonlinear equations, for which no systematic tool exists to analyze the feasible region. In the authors' prior work [14], by exploiting the connection between the feasibility of IA and the linear independence of the first order terms of IA constraints, an algebraic framework was established which gives a sufficient condition of IA feasibility. However, this framework is incomplete as it does not characterize necessary feasibility conditions.
- **Design GIA transceivers under general configuration**: For networks with a general configuration, existing IA transceiver design algorithms may not be able to find a solution even when IA is feasible. The IA transceiver design problem is usually formulated into optimization problems [18]–[21]. However, these problems are non-convex,

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L. Ruan and M. Z. Win are with the Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: lruan@mit.edu; moewin@mit.edu).

V. K. N. Lau is with the Electrical and Computer Engineering Department, Hong Kong University of Science and Technology (HKUST), Clear Water Bay, Hong Kong (e-mail: eeknlau@ust.hk).

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TABLE I
APPLICABLE CONFIGURATIONS OF EXISTING NECESSARY AND SUFFICIENT IA FEASIBILITY CONDITIONS

Reference	Network Configuration
[9]	$K \in \mathbb{N}, \quad d_k = 1, \quad \forall k$
[10]	$K \geq 3, \quad d_k = d, \quad N_k^{(\ell)} = M_k = N, \quad \forall k$
[11]	$K \in \mathbb{N}, \quad d_k = d, \quad d \vert N_k^{(\ell)}$, and $d \vert M_k, \quad \forall k$
[12]	$K = 3, \quad d_k = d, \quad N_k^{(\ell)} = N, M_k = M, \quad \forall k$
[14]	$K \in \mathbb{N}, \quad d_k = d, \quad N_k^{(\ell)} = N, M_k = M$, $\min\{M, N\} \geq 2d, \quad \forall k$ (extension of [10]) $K \in \mathbb{N}, \quad d_k = d, \quad d \vert N_k^{(\ell)}$, or $d \vert M_k, \quad \forall k$ (extension of [9], [11])

TABLE II
APPLICABLE CONFIGURATIONS OF EXISTING IA ALGORITHMS

Reference	Type	Network Configuration
[4]	constructive	$K = 3, \; d_k = d, \; N_k^{(\ell)} = M_k = N, \forall k$
[12]	constructive	$K = 3, \; d_k = d, \; N_k^{(\ell)} = N, \; M_k = M, \forall k$
[15]	constructive	$K \in \mathbb{N}, \; d_k = 1, \; N_k^{(\ell)} = 2, \forall k$
[16]	constructive	$K \geq 2, \; d_k = 1, \; N_k^{(\ell)} = M_k = K - 1, \forall k$
[18]–[20]	iterative	general configuration

making it challenging to find solutions. Moreover, in a network with many nodes, the dimension of the transceiver matrices is large. Solving a non-convex, high-dimensional problem is difficult.

B. Contribution of This Work

In this work, we consider MIMO wireless-tap networks[1] with LJs. To address the challenge in GIA feasibility analysis, we have established a theoretical framework by employing tools from algebraic geometry [22]. This framework shows the (almost sure) equivalence of the feasibility of the GIA transceiver design problem, the algebraic independence of GIA constraints, and the linear independence of the first order terms of GIA constraints, and hence enables us to propose and prove a necessary and sufficient GIA feasibility condition for networks with general configurations. By combining this condition with graph theory, we characterize the relationship between network configuration and GIA feasibility.

To address the challenge in GIA transceiver design, we exploit the equivalence between algebraic independence of GIA constraints and full rankness of their Jacobian matrix, and prove that when GIA is feasible, in a set of corresponding interference minimization problems, there is no performance gap between local and global optimums. This fact enables us to find solutions for the GIA transceiver design problem by adopting existing local search algorithms. The feasibility analysis and transceiver design for GIA covers those for IA as a special case.

C. Notations

1) General: a, $\mathbf{a}$, $\mathbf{A}$, and $\mathcal{A}$ represent scalar, vector, matrix, and set/space, respectively. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, real numbers, and complex numbers, respectively.

2) Functions: $n|m$ denotes that n divides m, and $n \bmod m$ denotes n modulo m, $n, m \in \mathbb{Z}$. $\mathbb{I}\{\cdot\}$ is the indicator function. $\binom{n}{m}$ is the Binomial coefficient with parameters $n, m \in \mathbb{N}$. $|a|$ represents the absolute value of scalar a, and $|\mathcal{A}|$ represents the cardinality of set $\mathcal{A}$. $f(\mathcal{A})$ denotes a function of all the elements in set $\mathcal{A}$.

3) Linear Algebra: The operators $(\cdot)^{\mathrm{T}}$, $(\cdot)^{\dagger}$, $\det(\cdot)$, $\mathrm{Rn}(\cdot)$, $\|\cdot\|_{\mathrm{F}}$, $\mathrm{Tr}(\cdot)$, $(\cdot)^{\sharp}$, $\mathcal{N}(\cdot)$, and $\mathrm{Vec}(\cdot)$ denote transpose, Hermitian transpose, determinant, rank, Frobenius norm, trace, Moore--Penrose pseudo inverse, null space, and vectorization of a matrix. $\mathrm{span}(\mathbf{A})$ and $\mathrm{span}(\{\mathbf{a}\})$ denote the linear space spanned by the column vectors of $\mathbf{A}$ and the vectors in set $\{\mathbf{a}\}$, respectively. $\dim(\cdot)$ denotes the dimension of a space. $\mathrm{diag}_n(\mathbf{A}, \ldots, \mathbf{X})$ represents a block diagonal matrix with submatrices $\mathbf{A}, \ldots, \mathbf{X}$ on its n-th diagonal. For instance,

$$\mathrm{diag}_{-1}([2,1],[1,2]) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}.$$

$\mathrm{diag}(\mathbf{A}, \ldots, \mathbf{X}) = \mathrm{diag}_0(\mathbf{A}, \ldots, \mathbf{X})$, and $\mathrm{diag}[m](\mathbf{A}) = \mathrm{diag}(\underbrace{\mathbf{A}, \ldots, \mathbf{A}}_{m \text{ times}})$.

4) Algebraic Geometry: For a field $\mathcal{K}$, $\mathcal{K}(x_1, x_2 \ldots, x_S)$ represents the field of rational functions in variables $x_1, x_2, \ldots, x_S$ with coefficients drawn from $\mathcal{K}$. Notation $\langle f_1, f_2, \ldots, f_L \rangle$ denotes the ideal generated by polynomials $f_1, f_2 \ldots, f_L$; notation $\mathcal{V}(\cdot)$ denotes vanishing set of an ideal; and notation $\mathbf{J}_{\mathbf{x}}(f_1, f_2 \ldots, f_L)$ represents the Jacobian matrix of polynomials $f_1, f_2, \ldots, f_L \in \mathcal{K}(x_1, x_2 \ldots, x_S)$ evaluated at point $\mathbf{x} \in \mathcal{K}^S$.

II. PROBLEM FORMULATION

In this section, the system model of wireless-tap networks is described, which is a generalization of interference networks, and then the GIA transceiver design problem is formulated.

A. System Model

Consider a network consisting of K legitimate transmitter (LT)-LR pairs, J LJs and K ERs,[2] as illustrated in Fig. 1. (Note that LTs and LJs are indexed from 1 to K and from $K+1$ to $K+J$, respectively.) Suppose LT j (or LJ j, if $j > K$), LR k, and ER k are equipped with M_j, $N_k^{(\ell)}$, and $N_k^{(e)}$ antennas, respectively. At each time slot, LT (or LJ) j sends d_j independent symbols. LT k attempts to send confidential messages to LR k, while ER k attempts to intercept these messages. LJ j transmits dummy data to generate interference.

The received signals $\mathbf{y}_k^{(\ell)}, \mathbf{y}_k^{(e)} \in \mathbb{C}^{d_k}$ at LR k and ER k are given by

$$\mathbf{y}_k^{(\iota)} = (\mathbf{U}_k^{(\iota)})^{\dagger} \Big(\mathbf{H}_{kk}^{(\iota)} \mathbf{V}_k \mathbf{x}_k + \sum_{j=1, \neq k}^{\tilde{K}} \mathbf{H}_{kj}^{(\iota)} \mathbf{V}_j \mathbf{x}_j + \mathbf{z}_k^{(\iota)} \Big), \quad (1)$$

where $\tilde{K} = K + J$, $\mathbf{H}_{kj}^{(\iota)} \in \mathbb{C}^{N_k^{(\iota)} \times M_j}$, $\iota \in \{\ell, e\}$ are the channel matrices from LT (or LJ) j to LR k or ER k, whose en-

[1]"wireless wiretap" is referred to as "wireless-tap" in this paper to emphasize the wireless nature of the propagation medium.

[2]In fact, as the proposed GIA technique does not require the channel state of the eavesdropping network, the ERs are not involved in GIA feasibility analysis. However, they remain in the system model to make the notation consistent with Part II.

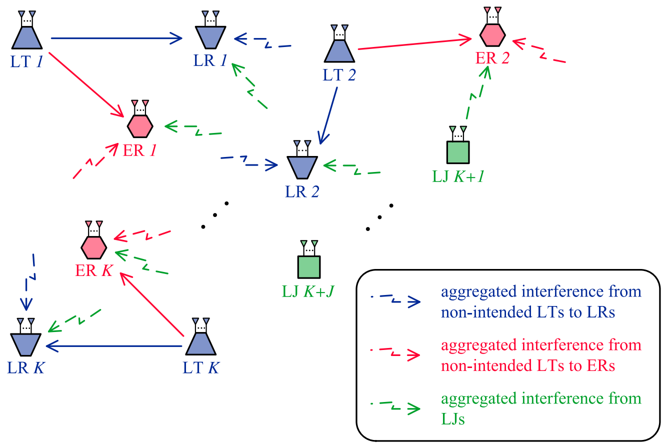

Fig. 1. Network configuration of wireless-tap networks with LJs.

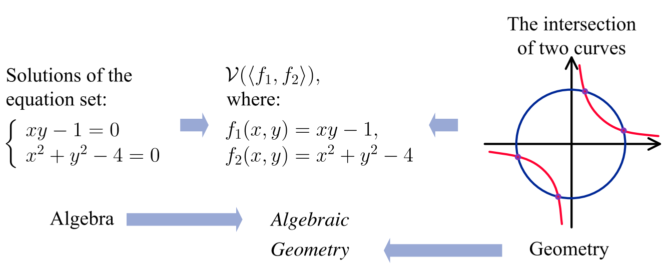

Fig. 2. An illustration of the correspondence of Algebra, Geometry and *Algebraic geometry*, where $\mathcal{V}(\langle f_1, f_2\rangle)$ denotes the vanishing set of the ideal generated by $\langle f_1, f_2\rangle$ [23, Def. 1, Section 1.4].

tries are independent random variables drawn from continuous distributions; $\mathbf{x}_j \in \mathbb{C}^{d_j}$ is the encoded information symbol at LT (or LJ) j; $\mathbf{V}_j \in \mathbb{C}^{M_j \times d_j}$ is the precoder at LT (or LJ) j; $\mathbf{U}_k^{(\iota)} \in \mathbb{C}^{N_k^{(\iota)} \times d_k}$, $\iota \in \{\ell, e\}$ is the decoder at LR k or ER k; and $\mathbf{z}_k^{(\iota)} \in \mathbb{C}^{N_k^{(\iota)} \times 1}$, $\iota \in \{\ell, e\}$ is the white Gaussian noise with zero mean and unit variance. The transmission power of LT (or LJ) j is given by

$$P_j = \mathbb{E}\left\{\mathrm{Tr}\left(\mathbf{x}_j^\dagger \mathbf{V}_j^\dagger \mathbf{V}_j \mathbf{x}_j\right)\right\}. \tag{2}$$

Define the configuration of the legitimate network as $\chi \triangleq \{(M_1, M_2, \ldots, M_{\tilde{K}}), (N_1^{(\ell)}, N_2^{(\ell)}, \ldots, N_K^{(\ell)}), (d_1, d_2, \ldots, d_{\tilde{K}})\}$.

Remark 2.1 (Applicability to Interference Networks): The wireless-tap network proposed above is a generalization of interference networks. Specifically, when there is no LJ, i.e., $\tilde{K} = K$, and the channel state of the eavesdropping links are zero matrices, i.e., $\mathbf{H}_{kj}^{(\ell)} = \mathbf{0}$, $\forall k, j$, the channel model (1) is reduced to that of conventional MIMO interference networks. Hence, as further illustrated in Remark 2.2, the theoretical results obtained in this work apply to MIMO interference networks. □

B. GIA Transceiver Design With Flexible Alignment Set

Classical IA requires the canceling of interference on all cross links. However, in large-scale networks this requirement may be infeasible and unnecessary. On one hand, the limited policy space in transceiver design may be insufficient to cancel interference on all cross links; on the other hand, some links may have very deep fading and hence there is no need to cancel interference on these links. Hence, to develop GIA strategies that fit large-scale networks, a more flexible approach must be adopted, in which the legitimate partners selectively cancel interference on a subset of cross links. This problem is formulated as follows:

Problem 2.1 (GIA Transceiver Design): Design transceivers $\{\mathbf{U}_k^{(\ell)}, \mathbf{V}_j\}$, $k \in \{1, 2, \ldots, K\}$, $j \in \{1, 2, \ldots, \tilde{K}\}$ which satisfy the following constraints:

$$\mathrm{Rn}\left((\mathbf{U}_k^{(\ell)})^\dagger \mathbf{H}_{kk}^{(\ell)} \mathbf{V}_k\right) = d_k,\ \forall k \in \{1, 2, \ldots, K\}, \tag{3}$$

$$\mathrm{Rn}(\mathbf{V}_j) = d_j,\ \forall j \in \{K+1, K+2, \ldots, \tilde{K}\}, \tag{4}$$

and

$$(\mathbf{U}_k^{(\ell)})^\dagger \mathbf{H}_{kj}^{(\ell)} \mathbf{V}_j = \mathbf{0},\ \forall (k, j) \in \mathcal{A} \tag{5}$$

where

$$\mathcal{A} \subseteq \mathcal{A}_{\mathrm{all}} = \{(k, j) : k \in \{1, 2, \ldots, K\}, j \in \{1, 2, \ldots, \tilde{K}\}, k \neq j\}$$

is the alignment set. It characterizes the set of cross links on which interference is to be canceled. □

Remark 2.2 (Connection Between IA and GIA Problems): When there are no LJs and the alignment set includes all cross links, i.e., $\tilde{K} = K$, $\mathcal{A} = \mathcal{A}_{\mathrm{all}}$, Problem 2.1 is converted to the classical IA problem on MIMO interference networks [9] (without symbol extension). □

Table I and Table II outline the contribution of existing works on IA (i.e., with $\tilde{K} = K$, $\mathcal{A} = \mathcal{A}_{\mathrm{all}}$) feasibility analysis and transceiver design. From these tables, it can be seen that IA feasibility conditions are determined for special configurations, and constructive IA transceiver design algorithms are also only applicable to special cases. Although existing iterative IA transceiver design algorithms apply to general configurations, they may not converge to a global optimum. In other words, the outputs of iterative algorithms may not be solutions of the IA problem. In this paper, we will determine the GIA feasibility conditions and develop algorithms that solve GIA problems for networks with general configuration and alignment sets.

III. Preliminaries

In this section, the mathematical approaches adopted in the existing theoretical works is outlined. Then the notion of algebraic independence is introduced, which is the most important mathematical concept adopted in this work.

A. Challenge in IA Feasibility Analysis

There is an inherent connection between the feasibility of a set of polynomial equations and algebraic geometry [24], as illustrated in Fig. 2. As a result, several prior works on IA feasibility analysis convert the IA problem into a polynomial form and then adopt tools from algebraic geometry. In fact, Problem 2.1 can be converted to the following form:[3]

Problem 3.1 (Polynomial Form of GIA Transceiver Design): Design $\tilde{\mathbf{U}}_k \in \mathbb{C}^{(N_k^{(\ell)} - d_k) \times d_k}$, $\tilde{\mathbf{V}}_j \in \mathbb{C}^{(M_j - d_j) \times d_j}$ such that

$$\begin{aligned} f_{kjpq}(\{\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j\}) \triangleq\ & \tilde{\mathbf{u}}_k^\dagger(p) \mathbf{H}_{kj}^{(\ell)}(d_k + 1 : N_k^{(\ell)}, q) \\ & + \mathbf{H}_{kj}^{(\ell)}(p, d_j + 1 : M_j) \tilde{\mathbf{v}}_j(q) \\ & + \tilde{\mathbf{u}}_k^\dagger(p) \mathbf{H}_{kj}^{(\ell)}(d_k + 1 : N_k^{(\ell)}, d_j + 1 : M_j) \\ & \times \tilde{\mathbf{v}}_j(q) \\ =\ & -h_{kj}(p, q) \end{aligned} \tag{6}$$

[3]This statement will be proved formally in Theorem 4.1.

where $k, j \in \mathcal{A}, p \in \{1, 2, \ldots, d_k\}, q \in \{1, 2, \ldots, d_j\}$, $\tilde{\mathbf{u}}_k(q)$, and $\tilde{\mathbf{v}}_j(q)$ represent the q-th column of $\tilde{\mathbf{U}}_k$ and $\tilde{\mathbf{V}}_j$, respectively. $h_{kj}(p, q)$ is the element in the p-th row and q-th column of $\mathbf{H}_{kj}^{(\ell)}$, and $\mathbf{H}_{kj}^{(\ell)}(p : p', q : q')$ represents the submatrix intersected by p to p'-th rows and q to q'-th columns of $\mathbf{H}_{kj}^{(\ell)}$. □

Challenge of Nonlinearity

In the polynomials f_{kjpq} defined above, there are second order terms, i.e., $\tilde{\mathbf{u}}_k^{\dagger}(p)\mathbf{H}_{kj}^{(\ell)}(d_k + 1 : N_k^{(\ell)}, d_j + 1 : M_j)\tilde{\mathbf{v}}_j(q)$. The presence of these second order terms makes it difficult to analyze the feasible region of Problem 3.1. This is because there are very few systematic tools that address the solvability issue of a set of nonlinear polynomial equations.

B. Challenge in IA Transceiver Design

Existing IA transceiver design algorithms can be classified into two categories: constructive algorithms and iterative algorithms. The constructive algorithms design transceivers according to some closed-form functions of the channel states. However, as illustrated in Table II, these algorithms only apply to limited configurations.

Iterative algorithms are applicable to networks with a general configuration. The most influential iterative algorithm was proposed in [18] and [19].[4] This algorithm searches for the IA solution by exploiting the uplink and downlink reciprocity and alternatively updates precoders and decoders in the following problem.

Problem 3.2 (Interference Minimization):

$$\underset{\mathbf{V}_j, \mathbf{U}_k^{(\ell)}}{\text{minimize}} \sum_{k=1}^{K} \sum_{j=1, \neq k}^{K} \frac{P_j}{d_j} \mathrm{Tr}\Big(\mathbf{V}_j^{\dagger}\mathbf{H}_{kj}^{\dagger}\mathbf{U}_k^{(\ell)}(\mathbf{U}_k^{(\ell)})^{\dagger}\mathbf{H}_{kj}\mathbf{V}_j\Big) \quad (7)$$

$$\text{subject to } \mathbf{V}_j^{\dagger}\mathbf{V}_j = \mathbf{I}, \qquad (\mathbf{U}_k^{(\ell)})^{\dagger}\mathbf{U}_k^{(\ell)} = \mathbf{I}, \quad \forall k, j. \quad (8)$$

Although widely adopted in the literature, the alternative minimization algorithm converges to a local optimum. In other words, it may not be able to cancel all interference even in IA feasible regions.[5] The convergence issue is challenging because of the non-convexity challenge elaborated below.

Challenge of Non-convexity

(1) The objective function (7) is not a convex function of the optimization variables $\mathbf{V}_j, \mathbf{U}_k^{(\ell)}$;

(2) The policy space defined by (8) is non-convex.

C. Introduction to Algebraic Independence

To overcome the nonlinearity and non-convexity challenges in the IA problem, a theoretical framework will be developed based on one of the key notions in algebraic geometry, i.e., *algebraic independence*. In this section, the definition of algebraic independence will be introduced and intuitions associated with the notion will be highlighted.

First recall linear independence. Let $\mathcal{K}$ be a field, then the standard definition of linear independence is given by:

Definition 1 (Linear Independence—Form I): A set of vectors $\mathbf{a}_l \in \mathcal{K}^S$, $l \in \{1, 2, \ldots, L\}$ are linearly independent iff $\sum_{l=1}^{L} k_l \mathbf{a}_l \neq \mathbf{0}, \forall [k_1, \ldots, k_L] \neq \mathbf{0}, \in \mathcal{K}^L$. □

In fact, Definition 1 can be transformed to the following equivalent form, which involves linear functions:

Definition 2 (Linear Independence—Form II): Define linear functions $f_l = \sum_{s=1}^{S} a_l(s)x_s$, $l \in \{1, 2, \ldots, L\}$, where $a_l(s)$ is the s-th element of $\mathbf{a}_l$. Coefficient vectors $\{\mathbf{a}_l\}$ are linearly independent iff $G(f_1, \ldots, f_L) \not\equiv 0$, $\forall$ non-zero linear function G. □

With Definition 2, we are ready to introduce algebraic independence. In fact, one just need to replace "linear function" by "polynomial" in Definition 2 to arrive at the definition for algebraic independence:

Definition 3 (Algebraic Independence): Polynomials $f_l \in \mathcal{K}(x_1, \ldots, x_S)$, $l \in \{1, 2, \ldots, L\}$, are algebraically independent iff $G(f_1, \ldots, f_L) \not\equiv 0$, $\forall$ non-zero polynomial $G \in \mathcal{K}(z_1, \ldots, z_L)$. □

Remark 3.1 (Linear and Algebraic Independence): The underlined parts in Definition 2 and 3 highlight that algebraic independence is an extension of linear independence. In the light of this information, it is reasonable to guess the properties of algebraic independence based on those of linear independence. For instance, if a statement holds conditional on linear independence, it is possible that a similar statement also holds conditional on algebraic independence. As will be illustrated in Remark 4.1, this intuition does help to construct a unified algebraic framework for both GIA feasibility analysis and algorithm design. □

IV. Feasibility Conditions and Transceiver Design

In this section, the main theoretical results on the GIA feasibility analysis and transceiver design are proposed and proved. First, an algebraic framework is established, which shows the (almost sure) equivalence of 1) feasibility of Problem 2.1, 2) algebraic independence of $\{f_{kjpq}\}$ defined in (6), 3) linear independence of the coefficient vectors of the first order terms in $\{f_{kjpq}\}$, and 4) full rankness of the Jacobian matrix of $\{f_{kjpq}\}$. Based on this framework, a necessary and sufficient feasibility condition of the GIA problem and design algorithms will be given to solve the GIA problem.

A. Mathematical Framework

We will first define the coefficient matrix of the first order terms of GIA constraints, then list the three theorems that construct the algebraic framework outlined in Fig. 3, and finally elaborate the intuition of these theorems by showing their counterparts in linear algebra.

Define $\mathbf{H}_{\text{all}}$ as the matrix aggregated by the coefficient vectors of the first order terms in $\{f_{kjpq}\}$. The structure of $\mathbf{H}_{\text{all}}$ is described in Fig. 4, where the submatrices $\mathbf{H}_{kj}^{\mathrm{U}} \in \mathbb{C}^{(d_k d_j) \times (d_k(N_k^{(\ell)} - d_k))}$ and $\mathbf{H}_{kj}^{\mathrm{V}} \in \mathbb{C}^{(d_k d_j) \times (d_j(M_j - d_j))}$ are defined by (9)–(10) at the bottom of the next page, where $h_{kj}(p, q)$ denotes the element in the p-th row and q-th column of $\mathbf{H}_{kj}^{(\ell)}$, $k \neq j, k \in \{1, 2, \ldots, K\}, j \in \{1, 2, \ldots, \tilde{K}\}$. Note that the coefficient vectors of the first order terms in $\{f_{kjpq}\}$ are linearly independent iff $\mathbf{H}_{\text{all}}$ is full row-rank.

[4]There are some differences between the algorithms proposed in [18] and [19]. However, the structure and the idea of these two algorithms are similar.

[5]That having been said, from the extensive numerical tests in Section V, we tend to believe that the algorithm proposed in [18] converges to a *global* optimum when IA is feasible. However, this conjecture is not proved in the literature.

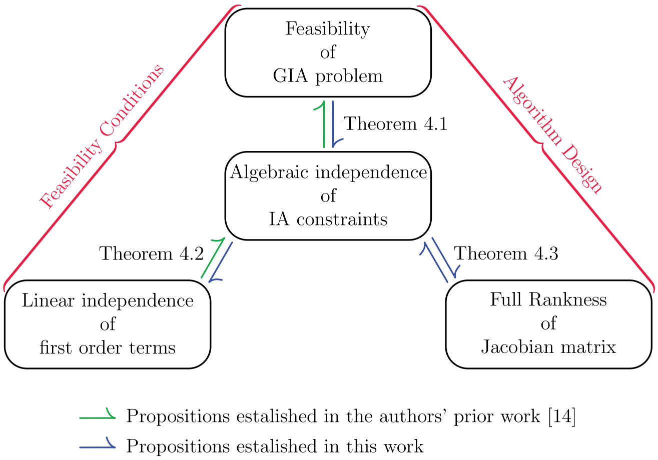

Fig. 3. Outline of the algebraic framework for the GIA problem.

The following three theorems construct the algebraic framework for GIA feasibility analysis and algorithm design.

Theorem 4.1 (Equivalence of Feasibility and Algebraic Independence): Under a network configuration χ, Problem 2.1 has solutions almost surely[6] iff the polynomials $\{f_{kjpq}\}$ defined in (6) are algebraically independent. The solution of Problem 2.1 can be obtained by first solving Problem 3.1 and then constructing transceivers $\{\mathbf{U}_k^{(\ell)}, \mathbf{V}_j\}$ via (11),

$$\mathbf{U}_k^{(\ell)} = \begin{bmatrix} \mathbf{I}_{d_k \times d_k} \\ \tilde{\mathbf{U}}_k \end{bmatrix}, \quad \mathbf{V}_j = \begin{bmatrix} \mathbf{I}_{d_j \times d_j} \\ \tilde{\mathbf{V}}_j \end{bmatrix}. \tag{11}$$

Proof: Please refer to Appendix A for the proof. □

Theorem 4.2 (Equivalence of Algebraic Independence and Linear Independence): Under a network configuration χ, matrix $\mathbf{H}_{\text{all}}$ (defined in Fig. 4) is either almost surely full row-rank or always row-rank deficient. In the first case, the polynomials $\{f_{kjpq}\}$ defined in (6) are almost surely algebraically independent. Otherwise, $\{f_{kjpq}\}$ are algebraically dependent.

Proof: Please refer to Appendix B for the proof. □

Theorem 4.3 (Equivalence of Algebraic Independence and Nonsingularity of Jacobian Matrix): The polynomials $\{f_{kjpq}\}$ defined in (6) are algebraically independent iff the Jacobian matrix $\mathbf{J}_{\mathbf{x}}(\{f_{kjpq}\})$ is full row-rank on a dense and open subset of $\mathbb{C}^V$, where $V = \sum_{k=1}^{K} d_k(N_k^{(\ell)} - d_k) + \sum_{j=1}^{\tilde{K}} d_j(M_j - d_j)$.

Proof: Please refer to Appendix C for the proof. □

[6]In this paper, "almost surely" means "with probability 1."

Remark 4.1 (Intuition from Linear Independence): To interpret the algebraic framework outlined in Fig. 3, consider a set of linear functions

$$f_l(x_1, x_2, \ldots, x_S) = \sum_{s=1}^{S} a_l(s) x_s = \mathbf{a}_l \mathbf{x}, \quad l \in \{1, 2, \ldots, L\} \tag{12}$$

where coefficient vector $\mathbf{a}_l = [a_l(1), a_l(2), \ldots, a_l(S)] \in \mathbb{C}^S$ and variable vector $\mathbf{x} = [x_1, x_2 \ldots, x_S]^{\mathrm{T}} \in \mathbb{C}^S$. Define

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_L \end{bmatrix}.$$

From linear algebra, the following proposition holds:

Proposition 4.1 (Equivalence of Linear Independence and Feasibility): Consider a vector $\mathbf{b} = [b_1, b_2 \ldots, b_L]^{\mathrm{T}}$ whose elements are independent random variables drawn from continuous distribution. Then linear equation set $f_l = b_l, l \in \{1, 2, \ldots, L\}$, i.e., $\mathbf{A}\mathbf{x} = \mathbf{b}$ has solutions iff vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_L$ are linearly independent.

Furthermore, for any vector $\mathbf{x} \in \mathbb{C}^S$, the Jacobian matrix is

$$\mathbf{J}_{\mathbf{x}}(f_1, f_2 \ldots, f_L) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_S} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_L}{\partial x_1} & \cdots & \frac{\partial f_L}{\partial x_S} \end{bmatrix} = \mathbf{A}. \tag{13}$$

Hence, the following proposition is also true:

Proposition 4.2 (Equivalence of Linear Independence and Nonsingularity of Jacobian Matrix): The coefficient vectors of $f_1, f_2, \ldots, f_L$ are linearly independent iff the Jacobian matrix $\mathbf{J}_{\mathbf{x}}(f_1, f_2, \ldots, f_L)$ is full row-rank for any $\mathbf{x} \in \mathbb{C}^S$.

By comparing Proposition 4.1 and 4.2 with Theorem 4.1 and 4.3, it can be seen that linear independence and algebraic independence play a similar role in these statements. This fact fits the insight illustrated in Remark 3.1. Actually, in the authors' previous work [14, Lem. 3.1], it was shown that if the coefficient vectors of the first order terms of a set of polynomials are *linearly* independent, then these polynomials are *algebraically* independent. The inverse proposition of [14, Lem. 3.1] is not true for general polynomials. Yet, in this paper, by exploiting the special structure of the polynomials defined in (6), the inverse proposition for GIA problems has been proved and hence Theorem 4.2 is obtained. □

$$\mathbf{H}_{kj}^{\mathrm{U}} = \mathrm{diag}[d_k]\left(\begin{bmatrix} h_{kj}(d_k+1,1), & h_{kj}(d_k+2,1), & \cdots, & h_{kj}(N_k^{(\ell)},1) \\ h_{kj}(d_k+1,2), & h_{kj}(d_k+2,2), & \cdots, & h_{kj}(N_k^{(\ell)},2) \\ \vdots & \vdots & \ddots & \vdots \\ h_{kj}(d_k+1,d_j), & h_{kj}(d_k+2,d_j), & \cdots, & h_{kj}(N_k^{(\ell)},d_j) \end{bmatrix}\right) \tag{9}$$

$$\mathbf{H}_{kj}^{\mathrm{V}} = \begin{bmatrix} \mathrm{diag}[d_j]\,(h_{kj}(1,d_j+1), & h_{kj}(1,d_j+2), & \cdots, & h_{kj}(1,M_j)) \\ \mathrm{diag}[d_j]\,(h_{kj}(2,d_j+1), & h_{kj}(2,d_j+2), & \cdots, & h_{kj}(2,M_j)) \\ & \cdots\cdots & & \\ \mathrm{diag}[d_j]\,(h_{kj}(d_k,d_j+1), & h_{kj}(d_k,d_j+2), & \cdots, & h_{kj}(d_k,M_j)) \end{bmatrix} \tag{10}$$

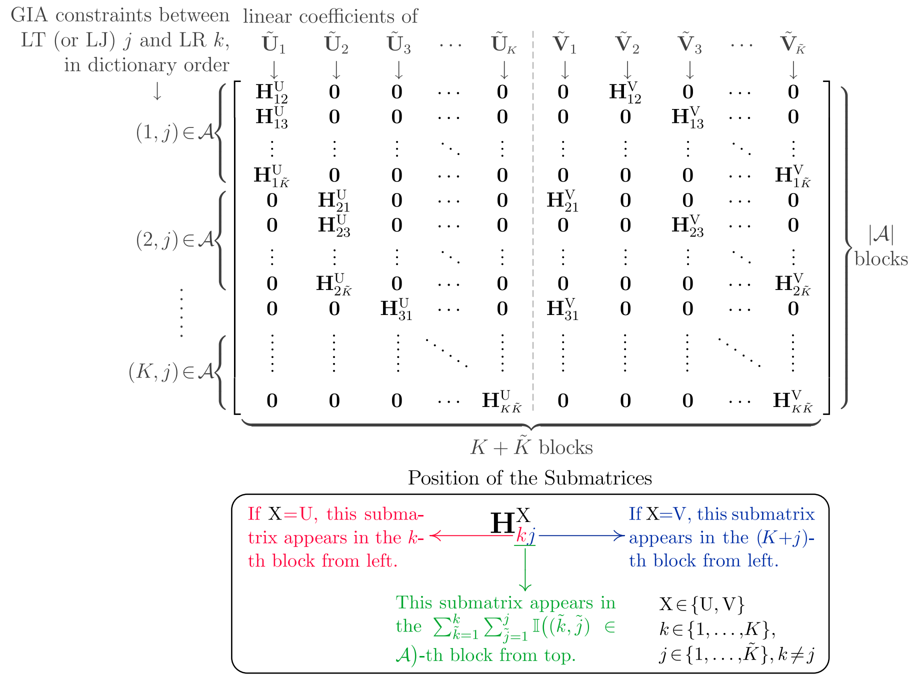

Fig. 4. The matrix scattered by the coefficient vectors of the linear terms in the polynomial form of the GIA constraints. For clear representation, $\mathcal{A}$ is equal to $\mathcal{A}_{\text{all}}$ in the figure. When $\mathcal{A} \subset \mathcal{A}_{\text{all}}$, part of the rows will not appear. The zero matrices which appear on the same block row with $\mathbf{H}_{kj}^{\text{U}}$ and $\mathbf{H}_{kj}^{\text{V}}$ have $d_k d_j$ rows. The zero matrices which appear on the same block column with $\mathbf{H}_{kj}^{\text{U}}$ or $\mathbf{H}_{kj}^{\text{V}}$ have $d_k(N_k - d_k^{(\ell)})$ and $d_j(M_j - d_j)$ columns, respectively.

B. Feasibility Conditions

Overcoming the Challenge of Nonlinearity

Based on the algebraic framework established in Section IV-A, we have the following theorem which determines the feasibility condition of GIA.

Theorem 4.4 (*Necessary and Sufficient Feasibility Condition*): Problem 2 has solutions almost surely iff matrix $\mathbf{H}_{\text{all}}$ in Fig. 4 is full row-rank.

Proof: This theorem is an immediate consequence of Theorems 4.1 and 4.2. □

With Theorem 4.4, there are three propositions illustrating the general trends on GIA feasibility.

Corollary 4.1 (Configuration and Alignment Set Dominate GIA Feasibility): Under given network configuration χ and alignment set $\mathcal{A}$, Problem 2.1 is either always infeasible or feasible almost surely.

Proof: This corollary is an immediate consequence of Theorem 4.4 and Lemma A.3. □

Corollary 4.2 (Scalability of GIA Feasibility): Under given alignment set $\mathcal{A}$, scaling the legitimate network configuration does not affect the GIA feasibility state, i.e., networks with configuration $\chi = \{(cM_1, cM_2, \ldots, cM_{\tilde{K}}), (cN_1^{(\ell)}, cN_2^{(\ell)}, \ldots, cN_K^{(\ell)}), (cd_1, cd_2, \ldots, cd_{\tilde{K}})\}$, $\forall c \in \mathbb{N}$ are either all GIA feasible or all GIA infeasible.

Proof: The proof is similar to that of [14, Cor. 3.2]. □

Remark 4.2 (Contributions of Corollary 4.1, 4.2): Theorem 4.4 gives a complete characterization of the feasibility condition of GIA problems. However, the feasibility condition in Theorem 4.4 is complicated as it relates to network configuration χ, alignment set $\mathcal{A}$, as well as the instantaneous channel state $\{\mathbf{H}_{kj}^{(\ell)}\}$. Corollary 4.1 simplifies this condition by showing that with probability 1, the feasible state is determined by configuration χ and alignment set $\mathcal{A}$. Corollary 4.2 further simplifies this condition by showing that networks with configurations different by a factor share the same feasible state.

One application of the propositions is an efficient method to check GIA feasibility. To determine if a set of networks with configuration $\chi = \{(cM_1, cM_2, \ldots, cM_{\tilde{K}}), (cN_1^{(\ell)}, cN_2^{(\ell)}, \ldots, cN_K^{(\ell)}), (cd_1, cd_2, \ldots, cd_{\tilde{K}})\}, \forall c \in \mathbb{N}$ is GIA feasible or not: set $c = 1$, randomly generate one channel state, and check if $\mathbf{H}_{\text{all}}$ is full row-rank or not. □

Corollary 4.3 (Necessary GIA Feasibility Condition): A network with configuration χ and alignment set $\mathcal{A}$ is GIA feasible only if

$$\sum_{j:(k,j)\in\mathcal{A}_{\text{sub}}} d_j(M_j - d_j) + \sum_{k:(k,j)\in\mathcal{A}_{\text{sub}}} d_k(N_k^{(\ell)} - d_k) \geq \sum_{(k,j)\in\mathcal{A}_{\text{sub}}} d_k d_j, \quad \forall \mathcal{A}_{\text{sub}} \subseteq \mathcal{A}. \tag{14}$$

Proof: Denote $\mathbf{H}_{\text{sub}}$ as the submatrix of $\mathbf{H}_{\text{all}}$ that corresponds to $\mathcal{A}_{\text{sub}}$. $\mathbf{H}_{\text{sub}}$ has $\sum_{(k,j)\in\mathcal{A}_{\text{sub}}} d_k d_j$ rows and $\sum_{j:(k,j)\in\mathcal{A}_{\text{sub}}} d_j(M_j - d_j) + \sum_{k:(k,j)\in\mathcal{A}_{\text{sub}}} d_k(N_k^{(\ell)} - d_k)$ non-zero columns. Hence, when (14) does not hold for a certain $\mathcal{A}_{\text{sub}}$, the corresponding $\mathbf{H}_{\text{sub}}$ is row-rank deficient and so is $\mathbf{H}_{\text{all}}$. From Theorem 4.4, the network is infeasible. This completes the proof. □

Remark 4.3 (Properness and Feasibility): In the pioneering work on IA feasibility analysis [9], the authors conjecture that a MIMO interference network is IA feasible only if the network is proper; i.e., the number of variables in transceiver design is no more than the number of IA constraints. This conjecture was later confirmed by [10] and [11]. Corollary 4.3 shows that properness is still a necessary feasibility condition for GIA problems. □

In the following, two corollaries are given which reveal simple insights into how legitimate network configuration χ and alignment set $\mathcal{A}$ determine the GIA feasibility.

Corollary 4.4 (Symmetric Configuration): Consider networks in which

1) configuration χ is symmetric, i.e., $d_k = d$, $M_k = M$, and $N_k^{(\ell)} = N, \forall k \in \{1, 2, \ldots, K\}$, with $\min\{M, N\} \geq 2d$;

2) alignment set between the LRs and LTs is L-regular, i.e.,

$$\sum_{j=1}^{K} \mathbb{I}\{(k,j) \in \mathcal{A}\} = \sum_{k=1}^{K} \mathbb{I}((k,j) \in \mathcal{A}) = L,$$

$\forall k, j \in \{1, 2, \ldots, K\}$; and

3) each LJ chooses the proper number of LRs to coordinate with, i.e.,

$$\sum_{k=1}^{K} \mathbb{I}((k,j) \in \mathcal{A}) \leq \lfloor \frac{M_j - d_j}{d} \rfloor,$$

$\forall j \in \{K+1, K+2, \ldots, \tilde{K}\}$. In these networks, Problem 2.1 has solutions almost surely iff inequality (15) is true, where

$$M + N - (L+2)d \geq 0. \tag{15}$$

Proof: Please refer to Appendix D for the proof. □

Corollary 4.5 ("Divisible" Configuration): When the network configuration χ satisfies

1) $d_k = d, \forall k \in \{1, 2, \ldots, \tilde{K}\}$ and

2) $d | N_k^{(\ell)}, \forall k \in \{1, 2, \ldots, K\}$ or $d | M_k, \forall k \in \{1, 2, \ldots, \tilde{K}\}$, Problem 2.1 has solutions almost surely iff inequality (16) is satisfied, where

$$\sum_{\substack{j:(k,j)\in \\ \mathcal{A}_{\text{sub}}}} (M_j - d) + \sum_{\substack{k:(k,j)\in \\ \mathcal{A}_{\text{sub}}}} (N_k^{(\ell)} - d) \geq d|\mathcal{A}_{\text{sub}}|, \quad \forall \mathcal{A}_{\text{sub}} \subseteq \mathcal{A}. \tag{16}$$

Proof: Please refer to Appendix E for the proof. □

Remark 4.4 (Backward Compatibility to Existing Works): If one specify the GIA problem to the classical IA problem, i.e., sets $\tilde{K} = K$ and $\mathcal{A} = \mathcal{A}_{\text{all}}$, then Corollary 4.4 and 4.5 are reduced to [14, Cor. 3.3] and [14, Cor. 3.4], respectively. Hence, these results are consistent with prior theoretical results on IA feasibility. □

C. GIA Transceiver Design

As illustrated in Section III-B, IA transceiver design is challenging because neither the policy space nor the objective function of the interference minimization problem is convex. This challenge will be overcome in two steps. In the first step, transform the problem to an equivalent one with convex policy space. In the second step, prove that there is no performance gap between the local and global optimums. Hence, despite the fact that the objective function is non-convex, the problem can be solved by various local search algorithms.

Overcoming the Challenge of Non-convexity (Step 1)

In Problem 3.1, the policy space is given by $\prod_{k=1}^{K} \mathbb{C}^{(N_k^{(\ell)} - d_k) \times d_k} \cdot \prod_{j=1}^{\tilde{K}} \mathbb{C}^{(M_j - d_j) \times d_j}$, which is a convex set. Hence, the first step is achieved by Theorem 4.1.

Then, transform Problem 3.1 to the following optimization problem (Problem 4.1). Note that Problem 3.1 is solved iff there exists a solution to Problem 4.1 that satisfies $F(\{g_{kjpq}(\tilde{\mathbf{U}}_k^*, \tilde{\mathbf{V}}_j^*)\}) = 0$.

Problem 4.1 (Optimization Form of GIA Problem):

$$\underset{\substack{\tilde{\mathbf{U}}_k \in \mathbb{C}^{(N_k^{(\ell)} - d_k) \times d_k} \\ \tilde{\mathbf{V}}_j \in \mathbb{C}^{(M_j - d_j) \times d_j}}}{\text{minimize}} \quad F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) \tag{17}$$

where $g_{kjpq} = f_{kjpq} + h_{kj}(p,q)$, f_{kjpq} is defined in (6), $(k,j) \in \mathcal{A}$, $p \in \{1, 2, \ldots, d_k\}$, $q \in \{1, 2, \ldots, d_j\}$, and F is a nonnegative, convex and continuously differentiable function. $F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) = 0$ iff $g_{kjpq} = 0, \forall k, j, p, q$. □

Overcoming the Challenge of Non-convexity (Step 2)

Theorem 4.5 (No Gap between Local and Global Optimums): When the polynomial form of the GIA problem, i.e., Problem 3.1 is feasible, in Problem 4.1, every local optimum is globally optimal.

Proof: Please refer to Appendix F for the proof. □

Remark 4.5 (The Role of Nonsingular Jacobian Matrix): The full row-rankness of the Jacobian matrix $\mathbf{J}_{\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j}(\{g_{kjpq}\})$ plays a key role in the proof of Theorem 4.5. To see how it works, consider a polynomial map $G : \mathbb{C}^N \to \mathbb{C}^M$. At point $\mathbf{x}_0 \in \mathbb{C}^N$,

$$G(\mathbf{x}_0 + \Delta\mathbf{x}) = G(\mathbf{x}_0) + \mathbf{J}_{\mathbf{x}_0}(G)\Delta\mathbf{x} + \mathcal{O}\left(\|\Delta\mathbf{x}\|^2\right). \tag{18}$$

Consider a neighborhood of $\mathbf{x}_0$ with $\|\Delta\mathbf{x}\| \ll 1$ and suppose the Jacobian matrix $\mathbf{J}_{\mathbf{x}_0}(G)$ is full row rank. In this case, the third term on the right-hand-side of (18) can be ignored compared to the second term, and $\Delta G = G(\mathbf{x}_0 + \Delta\mathbf{x}) - G(\mathbf{x}_0) = \mathbf{J}_{\mathbf{x}_0}(G)\Delta\mathbf{x}$ can be *any vector* in the neighborhood of $\mathbf{0}$.

Cascade G with a convex function $F : \mathbb{C}^M \to \mathbb{R}$, and suppose $\mathbf{x}_0$ is a local optimum of $F(G(\mathbf{x}))$. Then from the definition of local optimum and the property of ΔG just obtained, $F(G(\mathbf{x}_0) + \Delta G) \geq F(G(\mathbf{x}_0))$ for any vector ΔG in the neighborhood of $\mathbf{0}$. This implies $\mathbf{y}_0 \triangleq G(\mathbf{x}_0)$ is a local optimum of F. Since F is convex, $F(\mathbf{y}_0)$ must also be a global optimum of F and therefore $\mathbf{x}_0$ is a global optimum of $F(G(\mathbf{x}))$.

For Theorem 4.5, there is a weaker condition on the nonsingularity of the Jacobian matrix, i.e., full row-rank on a dense open subset. Yet, by imposing a stronger condition on the form of F, i.e., being continuously differentiable, the proof can be completed. □

Remark 4.6 (Theoretical Basis for GIA Transceiver Design): As illustrated in Fig. 5, based on Theorem 4.5, one can generate a set of algorithms that solve the GIA transceiver design problem. Moreover, the freedom in designing the specific form of F and choosing local search algorithms can be exploited to improve algorithm performances, such as message overhead, convergence speed and throughput. Hence, Theorem 4.5 sets up a theoretical basis to design and improve GIA transceiver design algorithms. □

Remark 4.7 (Consistency with Existing Theoretical Result): In [25], the authors have noted that IA transceiver design is highly challenging, but "there might still exist a polynomial time algorithm that can solve the problem ... with high probability (e.g., for almost all channel coefficients)." Noting that the polynomial form of the GIA transceiver design problem, i.e., Problem 3.1 is equivalent to the original GIA transceiver design problem, i.e., Problem 2.1 for almost all channel coefficients, the algorithms outlined in Fig. 5 solve the original GIA transceiver design problem almost surely. In this sense, this result confirms the prediction made in [25]. □

As an illustration, one specific GIA transceiver design algorithm will be presented. Let $F(\{x_i\}) = \sum_i x_i x_i^\dagger$; then Problem 4.1 can be rewritten as the follows:

Problem 4.2 (Reformed Interference Minimization):

$$\begin{array}{ll} \underset{\tilde{\mathbf{V}}_j, \tilde{\mathbf{U}}_k}{\text{minimize}} & \sum_{k=1}^{K} \sum_{j:(k,j)\in\mathcal{A}} \|\mathbf{U}_k^\dagger \mathbf{H}_{kj} \mathbf{V}_j\|_{\text{F}}^2 \\ \text{subject to} & \text{Eq. (11).} \end{array} \tag{19}$$

The following algorithm solves Problem 4.2:

Algorithm 1 (GIA Transceiver Design)

- **Step 1 Initialization**: Randomly generate $\tilde{\mathbf{V}}_j$, $j \in \{1, 2, \ldots, \tilde{K}\}$.
- **Step 2 Minimize interference leakage at the receiver side**: At LR k, update $\tilde{\mathbf{U}}_k$:
$$\tilde{\mathbf{U}}_k = -(\mathbf{B}_k \mathbf{A}_k^\sharp)^\dagger, \tag{20}$$
where $\mathbf{X}_k = [\mathbf{X}_{kj_1}, \mathbf{X}_{kj_2}, \ldots, \mathbf{X}_{kj_T}]$, $\{j_1, j_2, \ldots, j_T\} = \{j : (k,j) \in \mathcal{A}\}$, $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}\}$,
$$\begin{aligned} \mathbf{A}_{kj} &= \mathbf{H}_{kj}(d_k + 1 : N_k^{(\ell)}, 1 : d_j) \\ &\quad + \mathbf{H}_{kj}(d_k + 1 : N_k^{(\ell)}, d_j + 1 : M_j)\tilde{\mathbf{V}}_j, \text{ and} \\ \mathbf{B}_{kj} &= \mathbf{H}_{kj}(1 : d_k, 1 : d_j) + \mathbf{H}_{kj}(1 : d_k, d_j + 1 : M_j)\tilde{\mathbf{V}}_j. \end{aligned}$$
- **Step 3 Minimize interference leakage at the transmitter side**: At LT (LJ) j, update $\tilde{\mathbf{V}}_j$:
$$\tilde{\mathbf{V}}_j = -\mathbf{C}_j^\sharp \mathbf{D}_j, \tag{21}$$
where $\mathbf{X}_{\bar{j}} = \begin{bmatrix} \mathbf{X}_{k_1 j} \\ \mathbf{X}_{k_2 j} \\ \vdots \\ \mathbf{X}_{k_R j} \end{bmatrix}$,
$\{k_1, k_2, \ldots, k_R\} = \{k : (k,j) \in \mathcal{A}\}$, $\mathbf{X} \in \{\mathbf{C}, \mathbf{D}\}$,
$$\begin{aligned} \mathbf{C}_{kj} &= \mathbf{H}_{kj}(1 : d_k, d_j + 1 : M_j) \\ &\quad + \tilde{\mathbf{U}}_k^\dagger \mathbf{H}_{kj}(d_k + 1 : N_k^{(\ell)}, d_j + 1 : M_j), \text{ and} \\ \mathbf{D}_{kj} &= \mathbf{H}_{kj}(1 : d_k, 1 : d_j) + \tilde{\mathbf{U}}_k^\dagger \mathbf{H}_{kj}(d_k + 1 : N_k^{(\ell)}, 1 : d_j). \end{aligned}$$
- Repeat Step 2 and 3 until $\tilde{\mathbf{V}}_j$ and $\tilde{\mathbf{U}}_k$ converge. Substitute in (11) and obtain $\{\mathbf{V}_j^*, \mathbf{U}_k^{(\ell)^*}\}$.

Corollary 4.6 (Convergence of Algorithm 1): Algorithm 1 always converges. When IA is feasible, $\{\mathbf{V}_j^*, \mathbf{U}_k^{(\ell)^*}\}$ is a solution of Problem 2.1 almost surely.

Proof: Please refer to Appendix G for the proof. □

Remark 4.8 (Execute Algorithm 1 Distributively): Similar to the classical iterative IA algorithm [18], [20], Algorithm 1 can be executed distributively. To achieve this, after Step 2, LR k needs to send the updated $\tilde{\mathbf{U}}_k$ to LTs (or LJs) with index $j : (k,j) \in \mathcal{A}$, and after Step 3, LT (or LJ) j needs to send the updated $\tilde{\mathbf{V}}_j$ to LRs with index $k : (k,j) \in \mathcal{A}$. □

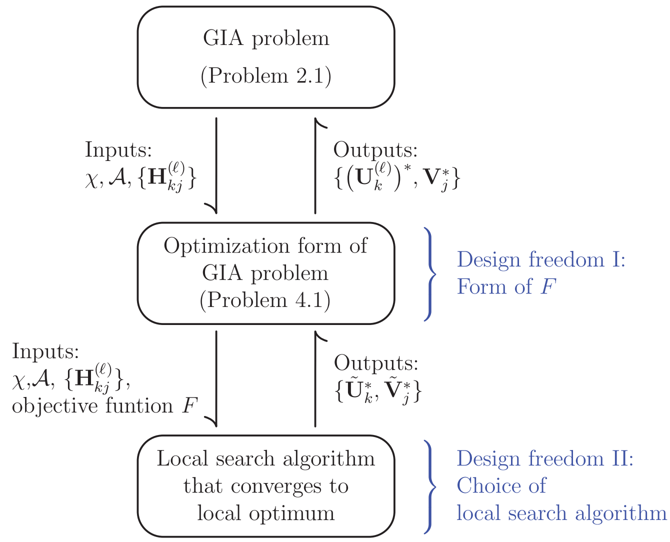

Fig. 5. Outline of GIA transceiver design algorithms based on Theorem 4.5.

V. Numerical Results

In this section, we will numerically test the convergence properties of the proposed algorithms, i.e., Algorithm 1 and the classical iterative IA algorithm proposed in [18]. Consider classical interference networks, i.e., networks with $\tilde{K} = K$, $\mathcal{A} = \mathcal{A}_{\text{all}}$. To verify if the IA algorithms can always find a solution in IA feasible scenarios, the following test is adopted.

Test 1 (Convergence Test on Random Interference Networks): Randomly select configuration within the set[7]

$$K \in \{3, 4, 5\}, \quad d_k \in 1, 2, 3, \quad d_k \le M_k, N_k^{(\ell)} \le 15, \quad \forall k$$

then randomly generate channel state $\{\mathbf{H}_{kj}^{(\ell)}\}$ following independent complex Gaussian distribution. First check if the network is IA feasible by testing full row-rankness of matrix $\mathbf{H}_{\text{all}}$ (defined in Fig. 4). If the network is IA feasible, perform the algorithm to be tested on this network. Denote the output transceivers after t rounds of iteration by $\{\mathbf{V}_k(t), \mathbf{U}_k^{(\ell)}(t)\}$. $\{\mathbf{V}_k(0), \mathbf{U}_k^{(\ell)}(0)\}$ are the initial guesses of the transceivers. Define the normalized power of interference (dB) after t rounds of iteration as

$$I(t) = 10 \log_{10} \frac{\sum_{k=1}^{K} \sum_{\substack{j=1 \\ j\neq k}}^{K} \left\|(\mathbf{U}_k^{(\ell)}(t))^\dagger \mathbf{H}_{kj}^{(\ell)} \mathbf{V}_j(t)\right\|_{\text{F}}^2}{\sum_{k=1}^{K} \sum_{\substack{j=1 \\ j\neq k}}^{K} \left\|(\mathbf{U}_k^{(\ell)}(0))^\dagger \mathbf{H}_{kj}^{(\ell)} \mathbf{V}_j(0)\right\|_{\text{F}}^2}. \tag{22}$$

If the normalized power of interference can be reduced below -60 dB after some t, the algorithm passes the test. Otherwise, if the algorithm converges to a point with $I(t) > -60$, it fails the test. □

Test 1 was performed for 10^6 times on both Algorithm 1 and the classical iterative IA algorithm. In all the IA feasible scenarios (about 6.6×10^5 cases), both algorithms pass the test. This result verifies the claim of Corollary 4.6.

To demonstrate how network configuration affects the convergence properties of the proposed algorithm and classical IA algorithm, consider three similar networks

- **Configuration 1** (Feasible Symmetric Network):

$$\chi = \{(6, 6, 6), (6, 6, 6), (3, 3, 3)\};$$

[7]The sizes of the networks are restricted so as to maintain manageable computation load.

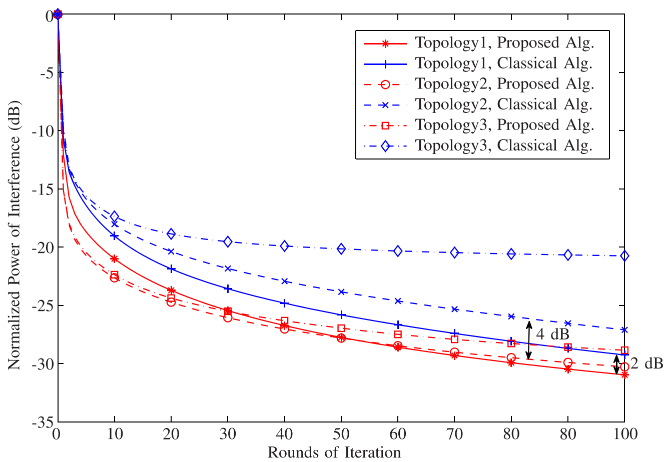

Fig. 6. Normalized power of interference as a function of rounds of iteration in different network configuration. For fair comparison, the output transceivers of both algorithms are scaled so that $\sum_{k=1}^{K} \mathrm{Tr}(\mathbf{V}_k^{\dagger}\mathbf{V}_k)$ and $\sum_{k=1}^{K} \mathrm{Tr}((\mathbf{U}_k^{(\ell)})^{\dagger}\mathbf{U}_k^{(\ell)})$ remain constant.

- **Configuration 2** (Feasible Asymmetric Network):

$$\chi = \{(5,5,5),(6,6,9),(3,3,3)\};$$

- **Configuration 3** (Infeasible Network):

$$\chi = \{(5,5,5),(5,7,9),(3,3,3)\}.$$

Fig. 6 illustrates the normalized power of interference $I(t)$ as a function of rounds of iteration t under the proposed and classical IA algorithms in the three network configurations. In the two IA feasible networks, both algorithms converge sub-linearly, with the proposed algorithm converging 2 dB and 4 dB faster in the symmetric and asymmetric cases respectively. In the IA infeasible network, under the classical IA algorithm, $I(t)$ converges to -21 dB, whereas the proposed algorithm reduces $I(t)$ to -28 dB after 100 rounds of iteration (and converges to -30 dB after 400 rounds).

VI. Summary

In Part I, we proposed a GIA approach to further improve the IA's capability in secrecy enhancement. As illustrated in Fig. 3, we established an algebraic framework that reveals the (almost sure) equivalence of 1) feasibility of GIA, 2) algebraic independence of GIA constraints, 3) linear independence of the coefficient vectors of the first order terms in GIA constraints, and 4) full rankness of the Jacobian matrix of GIA constraints. This framework allows us to address the two fundamental issues of GIA, i.e., feasibility conditions and transceiver design and hence sets up a theoretical foundation for GIA (and IA, as a special case) techniques.

Appendix A
Proof of Theorem 4.1

To prove the "if" side, first prove the following lemma.

Lemma A.1 (Algebraic Independence Leads to Solutions): $\{c_1, c_2, \ldots, c_L\} \in \mathbb{C}^L$ are independent random variables drawn from continuous distribution. Then if polynomials $f_l \in \mathbb{C}(x_1, x_2, \ldots, x_S)$, $l \in \{1, 2, \ldots, L\}$ are algebraically independent, equation set $f_l = c_l$, $l \in \{1, 2, \ldots, L\}$ has solutions almost surely. Otherwise, the equation set has no solution almost surely.

Proof: The first half of the lemma is proved in [14, Lem. 3.2]. Hence, the focus is on the second half of the lemma.

Denote $F : \mathbb{C}^S \rightarrow \mathbb{C}^L$ as the polynomial map defined by $\{f_l\}$. Since f_l are algebraically dependent, there exists a non-zero polynomial g such that $g(f_1, f_2, \ldots, f_L) \equiv 0$. Then for any point $[\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_L] \in F(\mathbb{C}^S)$, $g(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_L) = 0$. On the other hand, since g is a non-zero polynomial, and $\{c_1, c_2, \ldots, c_L\} \in \mathbb{C}^L$ are independent random variables drawn from continuous distribution, $g(c_1, c_2, \ldots, c_L) \neq 0$ almost surely. Hence, $[c_1, c_2, \ldots, c_L] \notin F(\mathbb{C}^S)$ almost surely. □

Now turn to the main flow of the proof of the "if" side. From Lemma A.1, when $\{f_{kjpq}\}$ are algebraically independent, Problem 3.1 has solutions almost surely. Then from (6), the solution $\{\mathbf{U}_k^{(\ell)}, \mathbf{V}_j\}$ constructed by (11) satisfies (5). Further noting that

- $\{\mathbf{U}_k, \mathbf{V}_j\}$ are functions of the channel state of the cross links $\{\mathbf{H}_{kj}^{(\ell)}, k \neq j\}$, and are hence independent of the channel state of the direct links $\{\mathbf{H}_{kk}^{(\ell)}\}$, and
- $\mathrm{Rn}(\mathbf{U}_k^{(\ell)}) = \mathrm{Rn}(\mathbf{V}_k) = d_k$,

we have that $\{\mathbf{U}_k, \mathbf{V}_j\}$ constructed by (11) satisfy (3) and (4) almost surely. Hence, in this case, Problem 2.1 has solutions almost surely.

The "only if" side will be proved by verifying its converse-negative proposition:

Proposition A.1: When $\{f_{kjpq}\}$ are algebraically dependent, Problem 2.1 has no solution almost surely.

To prove this proposition, first prove following lemmas.

Lemma A.2 (Algebraic Independence of Random Polynomials): The coefficients of polynomials $f_l \in \mathbb{C}(x_1, x_2, \ldots, x_S)$, $l \in \{1, 2, \ldots, L\}$ are random variables drawn from continuous distribution. Then polynomials $\{f_l\}$ are either always algebraically dependent or algebraically independent almost surely.

Proof: $\{f_l\}$ are algebraically dependent iff there exists a non-zero polynomial $g \in \mathbb{C}(y_1, y_2, \ldots, y_L)$ such that

$$g(f_1, f_2 \ldots, f_L) \equiv 0. \tag{23}$$

Without loss of generality, suppose g has N terms, whose coefficients are given by $\{c_1, c_2, \ldots, c_N\}$; then (23) can be rewritten as a set of linear equations:

$$\mathbf{Fc} = \mathbf{0}. \tag{24}$$

where $\mathbf{c} = [c_1, c_2 \ldots, c_N]^{\mathrm{T}}$, $\mathbf{F} \in \mathbb{C}^{S \times N}$, S is the number of terms in $g(f_1, f_2, \ldots, f_L)$ after combining like terms. For instance, suppose $f_1 = a_1 + b_1 x_1$, $f_2 = a_2 + b_2 x_2$ and $g = c_1 y_1 + c_2 y_2 + c_3 y_1 y_2$; then

$$\begin{aligned} g(f_1, f_2) = {} & [a_1, a_2, a_1 a_2]\mathbf{c} + [b_1, 0, b_1 a_2]\mathbf{c}x_1 \\ & + [0, b_2, a_1 b_2]\mathbf{c}x_2 + [0, 0, b_1 b_2]\mathbf{c}x_1 x_2. \end{aligned} \tag{25}$$

Hence, (24) is given by

$$\begin{bmatrix} a_1 & a_2 & a_1 a_2 \\ b_1 & 0 & b_1 a_2 \\ 0 & b_2 & a_1 b_2 \\ 0 & 0 & b_1 b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{26}$$

Note that (24) has non-zero solutions iff $\mathcal{N}(\mathbf{F}) \neq \{0\}$, i.e., $\mathbf{F}$ is column-rank deficient. From Lemma A.3, (24) either always has no non-zero solutions or has non-zero solutions almost surely. This completes the proof. □

Lemma A.3 (Rank of a Random Matrix): Suppose the entries of a matrix $\mathbf{F} \in \mathbb{C}^{M\times N}$ are either 0 or random variables drawn from continuous distribution. Then $\mathbf{F}$ is either always column-rank deficient or full column-rank almost surely.

Proof: If $M < N$, $\mathbf{F}$ is always column-rank deficient. Otherwise, denote all the $N \times N$ submatrices in $\mathbf{F}$ by $\{\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2, \ldots, \tilde{\mathbf{F}}_D\}$, where $D = \binom{M}{N}$; then $\mathbf{F}$ is full column-rank iff the determinant of at least one $\tilde{\mathbf{F}}_d$, $d \in \{1, 2, \ldots, D\}$ is not zero.

From the Leibniz formula [26, 6.1.1], the determinant $\tilde{\mathbf{F}}_d$ is given by a polynomial of the entries in $\tilde{\mathbf{F}}_d$. If this polynomial is a zero polynomial, the determinant of $\tilde{\mathbf{F}}_d$ is always 0. Otherwise, noting that the entries of $\tilde{\mathbf{F}}_d$ are drawn from continuous distribution, the value of this polynomial is non-zero almost surely. This completes the proof. □

Now turn to the main flow of the proof of Proposition A.1. Consider a solution $\{\mathbf{U}_k^{(\ell)^*}, \mathbf{V}_j^*\}$ of Problem 2.1. From (3), we have that $\mathrm{Rn}(\mathbf{U}_k^{(\ell)^*}) = d_k$ and $\mathrm{Rn}(\mathbf{V}_j^*) = d_j, \forall k, j$. Hence, every $\mathbf{U}_k^{(\ell)^*}$ (or $\mathbf{V}_j^*$) has at least d_k (or d_j) linearly independent row vectors. Denote the submatrices aggregated by these linearly independent rows by $\mathbf{U}_k^{(1)}$ (or $\mathbf{V}_k^{(1)}$). Transform $\mathbf{U}_k, \mathbf{V}_j$ as follows:

$$\mathbf{U}_k' = \mathbf{U}_k^{(\ell)}\left(\mathbf{U}_k^{(1)}\right)^{-1}, \quad \mathbf{V}_j' = \mathbf{V}_j\left(\mathbf{V}_j^{(1)}\right)^{-1}, \tag{27}$$

and let $\tilde{\mathbf{U}}_k$ and $\tilde{\mathbf{V}}_j$ be the nonconstant parts in $\mathbf{U}_k'$ and $\mathbf{V}_j'$, respectively. Then, $\{\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j\}$ satisfies a set of polynomial equations in the same form as (6), in which the position of $\{\mathbf{U}_k^{(1)}, \mathbf{V}_k^{(1)}\}$ in $\{\mathbf{U}_k', \mathbf{V}_j'\}$ only affects the indices of the coefficients. For example, suppose $\mathbf{U}_k^{(1)}$, $\mathbf{V}_j^{(1)}$ are given by the last $d_k \times d_k$ and $d_j \times d_j$ submatrices in $\mathbf{U}_k^{(\ell)}$ and $\mathbf{V}_j$ respectively; then (5) can be rewritten as

$$\begin{aligned} f_{kjpq} \triangleq\, & \tilde{\mathbf{u}}_k^\dagger(p)\mathbf{H}_{kj}^{(\ell)}(1 : N_k^{(\ell)} - d_k, M_j - d_j + q) \\ & + \mathbf{H}_{kj}^{(\ell)}(N_k^{(\ell)} - d_k + p, 1 : M_j - d_j)\tilde{\mathbf{v}}_j(q) \\ & + \tilde{\mathbf{u}}_k^\dagger(p)\mathbf{H}_{kj}^{(\ell)}(1 : N_k^{(\ell)} - d_k, 1 : M_j - d_j)\tilde{\mathbf{v}}_j(q) \\ =\, & - h_{kj}(N_k^{(\ell)} - d_k + p, M_j - d_j + q), \end{aligned} \tag{28}$$

which is the same as (6), except for the indices of the coefficients.

Since all entries the of channel state matrices $\mathbf{H}_{kj}^{(\ell)}$ are independent random variables drawn from continuous distribution, we have that if Problem 3.1 has no solution almost surely, for every possible position of $\{\mathbf{U}_k^{(1)}, \mathbf{V}_k^{(1)}\}$, the corresponding equation set also has no solution almost surely. Hence, Problem 2.1 has no solution almost surely.

Appendix B
Proof of Theorem 4.2

The proof of the first statement in Theorem 4.2 is given by Lemma A.3.

If matrix $\mathbf{H}_{\mathrm{all}}$ is full row-rank almost surely, from [14, Lem. 3.1], polynomials $\{f_{kjpq}\}$ are algebraically independent almost surely. Hence, the focus is on the other case.

The size of matrix $\mathbf{H}_{\mathrm{all}}$ is $C \times V$, where

$$C = \sum_{k=1}^{K} \sum_{\substack{j=1,\\(k,j)\in\mathcal{A}}}^{\tilde{K}} d_k d_j \text{ and}$$

$$V = \sum_{k=1}^{K} d_k(N_k^{(\ell)} - d_k) + \sum_{j=1}^{\tilde{K}} d_j(M_j - d_j).$$

If matrix $\mathbf{H}_{\mathrm{all}}$ is always row-rank deficient, there are two possibilities:

- When $C > V$: Denote $\mathcal{V}$ as the set of all entries in $\tilde{\mathbf{U}}_k$ and $\tilde{\mathbf{V}}_j$, $k \in \{1, 2, \ldots, K\}$, $j \in \{1, 2, \ldots, \tilde{K}\}$. From [27, Cor. 5.7], the dimension of the field $\mathcal{V}]$ is V. On the other hand, the number of the polynomials in $\{f_{kjpq}\}$, i.e., C, is greater than V. Hence, from [27, Def. 5.3], $\{f_{kjpq}\}$ must be algebraically dependent.
- When $C \le V$: Denote all the $C \times C$ submatrices in $\mathbf{H}_{\mathrm{all}}$ by $\{\tilde{\mathbf{H}}_1, \ldots, \tilde{\mathbf{H}}_D\}$, where $D = \binom{V}{C}$. Since $\mathbf{H}_{\mathrm{all}}$ is always row-rank deficient,

$$\det(\tilde{\mathbf{H}}_d) \equiv 0 \tag{29}$$

for all $d \in \{1, 2, \ldots, D\}$ and all possible channel states $\{\mathbf{H}_{kj}^{(\ell)}\}$. From the Leibniz formula, $\det(\tilde{\mathbf{H}}_d)$ is given by a polynomial of the entries in $\tilde{\mathbf{H}}_d$. Denote this polynomial by $g_d(\{h_{kj}(p, q)\}) \triangleq \det(\tilde{\mathbf{H}}_d)$. Then from (29), g_d are zero polynomials for all $d \in \{1, 2, \ldots, D\}$.

Next, consider the Jacobian matrix of $\{f_{kjpq}\}$, i.e., $\mathbf{J}_{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j}(\{f_{kjpq}\})$. From (6), $\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{f_{kjpq}\})$ has the same structure as $\mathbf{H}_{\mathrm{all}}$, with the following differences:

$$\begin{cases} \text{In (9), } h_{kj}(p, q)\text{, is replaced by} \\ \qquad h_{kj}(p, q) + \mathbf{H}_{kj}^{(\ell)}(p, d_j + 1 : M_j)\tilde{\mathbf{v}}_j(q)\text{, where} \\ \qquad p \in \{d_k+1, d_k+2, \ldots, N_k^{(\ell)}\}, q \in \{1, \ldots, d_j\}; \\ \text{In (10), } h_{kj}(p, q)\text{, is replaced by} \\ \qquad h_{kj}(p, q) + \tilde{\mathbf{u}}_k^\dagger(p)\mathbf{H}_{kj}^{(\ell)}(d_k+1 : N_k^{(\ell)}, q)\text{, where} \\ \qquad p \in \{1, 2, \ldots, d_k\}, q \in \{d_j+1, d_j+2, \ldots, M_j\}. \end{cases} \tag{30}$$

Denote all the $C \times C$ submatrices in $\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{f_{kjpq}\})$ by $\{\tilde{\mathbf{J}}_1, \ldots, \tilde{\mathbf{J}}_D\}$. Define linear functions $\ell_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)$ as

$$\ell_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j) = \begin{cases} h_{kj}(p, q) + \mathbf{H}_{kj}^{(\ell)}(p, d_j + 1 : M_j)\tilde{\mathbf{v}}_j(q), \\ \quad \text{if} \quad p \in \{d_k + 1, d_k + 2, ,\ldots, N_k^{(\ell)}\}, \text{and} \\ \qquad\quad q \in \{1, 2, \ldots, d_j\}; \\ h_{kj}(p, q) + \tilde{\mathbf{u}}_k^\dagger(p)\mathbf{H}_{kj}^{(\ell)}(d_k + 1 : N_k^{(\ell)}, q), \\ \quad \text{if} \quad p \in \{1, 2, \ldots, d_k\}, \text{and} \\ \qquad\quad q \in \{d_j + 1, d_j + 2, \ldots, M_j\}. \end{cases} \tag{31}$$

Then from (30), there is a one to one correspondence between $\{h_{kj}(p, q)\}$ and $\{\ell_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}$. Therefore, $\det(\mathbf{J}_d)$ can be written as the cascade of g_d and $\{\ell_{kjpq}\}$, i.e.,

$$\det(\mathbf{J}_d) = g_d(\{\ell_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}), \; d \in \{1, 2, \ldots, D\}. \tag{32}$$

Since $\{g_d\}$ are zero polynomials, $\det(\mathbf{J}_d) \equiv 0$, $\forall d \in \{1, 2, \ldots, D\}$, which means that $\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{f_{kjpq}\})$ is always row-rank deficient. From [28, Thm. 2.3], $\{f_{kjpq}\}$ are algebraically dependent.

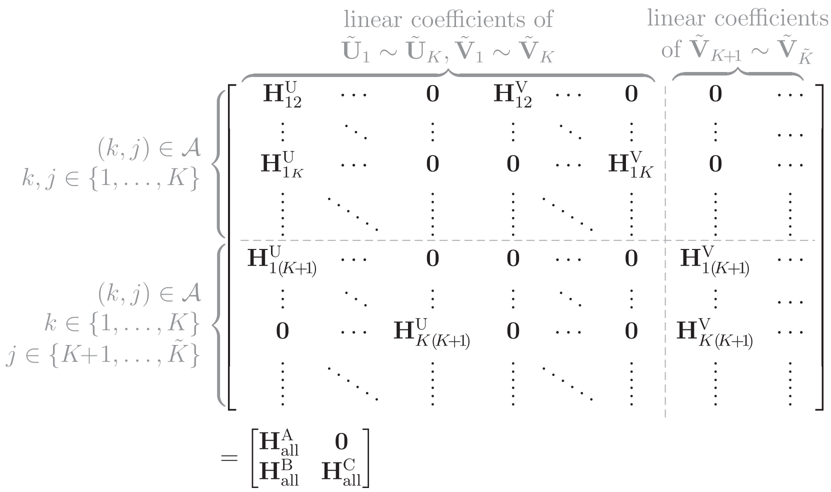

Fig. 7. Row-switching and separation of $\mathbf{H}_{\text{all}}$. For clear illustration, we have set $\mathcal{A} = \mathcal{A}_{\text{all}}$ when plotting the figure.

APPENDIX C
PROOF OF THEOREM 4.3

From [28, Thm. 2.2], when $\mathbf{J}_{\mathbf{x}}(\{f_{kjpq}\})$, is not always row-rank deficient, $\{f_{kjpq}\}$ are algebraically independent. Hence, the "if" side is proved.

The "only if" side is true if the following lemma holds:

Lemma C.1: If $\{f_{kjpq}\}$ are algebraically independent, $\mathbf{J}_{\mathbf{x}}(\{f_{kjpq}\})$ is row-rank deficient on a proper closed subset of $\mathbb{C}^V$.

From (32) and (31), the set in which $\mathbf{J}_{\mathbf{x}}(\{f_{kjpq}\})$ is row-rank deficient is given by $\cap_{d=1}^{D}\mathcal{N}_d$, where

$$\mathcal{N}_d \triangleq \Big\{\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j, k \in \{1,2,\ldots,K\}, j \in \{1,2,\ldots,\tilde{K}\} : \\ g_d(\{\ell_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) = 0\Big\} \tag{33}$$

If $g_d(\{\ell_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\})$ is a zero polynomial of $\{\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j\}$, $\mathcal{N}_d = \mathbb{C}^V$; otherwise, $\mathcal{N}_d$ is a proper closed set of $\mathbb{C}^V$. When $\{f_{kjpq}\}$ are algebraically independent, at least one $g_d(\{\ell_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\})$ is a non-zero polynomial. Noting that the intersection of closed sets is closed, Lemma C.1 is proved.

APPENDIX D
PROOF OF COROLLARY 4.4

From Theorem 4.4, one needs to show that $\mathbf{H}_{\text{all}}$ is full row rank iff (15) is true. As illustrated in Fig. 7, perform row switching and then separate $\mathbf{H}_{\text{all}}$ into four submatrices, i.e., $\mathbf{H}_{\text{all}}^{\text{A}} - \mathbf{H}_{\text{all}}^{\text{C}}$ and one zero matrix. The following lemma shows the full rankness of $\mathbf{H}_{\text{all}}^{\text{C}}$.

Lemma D.1 (Full rankness of $\mathbf{H}_{\text{all}}^{\text{C}}$): Under condition 3) in Corollary 4.4, $\mathbf{H}_{\text{all}}^{\text{C}}$ is full row rank almost surely.

Proof: Note that

$$\mathbf{H}_{\text{all}}^{\text{C}} = \text{diag}\left(\mathbf{H}_{K+1}^{\text{C}}, \ldots, \mathbf{H}_{\tilde{K}}^{\text{C}}\right) \tag{34}$$

where $\mathbf{H}_j^{\text{C}}, j \in \{K+1, \ldots, \tilde{K}\}$ is aggregated by submatrices $\mathbf{H}_{kj}^{\text{V}}, \forall k : (k,j) \in \mathcal{A}$. From the structure of $\mathbf{H}_{kj}^{\text{V}}$ in (10), by doing row switching operations, $\mathbf{H}_j^{\text{C}}$ can be transformed into a block diagonal matrix with d_j diagonal blocks. Note that

(a) the size of these diagonal blocks is

$$(d\sum_{k=1}^{K}\mathbb{I}\{(k,j) \in \mathcal{A}\}) \times (M_j - d_j);$$

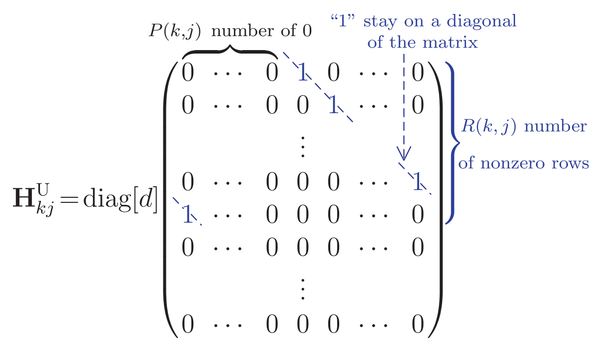

Fig. 8. Specify $\{\mathbf{H}_{kj}^{\text{U}}\}$.

(b) within each diagonal block, all entries are independent random variables.

Hence, when condition 3) in Corollary 4.4 holds, the diagonal blocks in $\mathbf{H}_j^{\text{C}}$ are full row-rank almost surely. Therefore, $\mathbf{H}_j^{\text{C}}$ is full row-rank almost surely. Substituting this result to (34), $\mathbf{H}_{\text{all}}^{\text{C}}$ is full row-rank almost surely. This completes the proof. □

With Lemma D.1, and further noting that $\mathbf{H}_{\text{all}}$ is a block-upper-triangular matrix, the corollary holds if the following proposition is true:

Proposition D.1: Under condition 1) and 2) in Corollary 4.4, $\mathbf{H}_{\text{all}}^{\text{A}}$ is full row rank iff (15) is true.

When (15) is not satisfied, $\mathbf{H}_{\text{all}}^{\text{A}}$ is row-rank deficient as it has more rows than columns. Hence, the "only if" statement in Proposition D.1 is proved. The "if" side can be proved via the following steps:

A. Construct one special category of channel state $\{\mathbf{H}_{kj}\}$.
B. Show that $\mathbf{H}_{\text{all}}$ is full rank almost surely under the special category of channel state.
C. From the first statement in Theorem 4.2, if Procedure B is completed, $\mathbf{H}_{\text{all}}^{\text{A}}$ is full rank almost surely, and this proves the corollary.

Construct a special $\mathbf{H}_{\text{all}}$ by using tools from graph theory. Consider a graph $\mathcal{G}$ whose vertexes are the nodes of the network and there is an edge between LT j and LR k, if $(k,j) \in \mathcal{A}$. Then from [29, Thm. 8.15], when the alignment set is L-regular, there is a proper L-edge-coloring [29, Page 138] for the graph. Denote the coloring of an edge between LT j and LR k by $f(k,j) \in \{1,2,\ldots,L\}$ and specify $\{\mathbf{H}_{kj}^{\mathrm{U}}\}$ as in Fig. 8, in which

$$P(k,j) = d(f(k,j)-1) \bmod (N-d) \tag{35}$$

$$R(k,j) = \begin{cases} d & \text{if } f(k,j) \le \lfloor \frac{N}{d} \rfloor, \\ (N-1) \bmod d & \text{if } f(k,j) = \lfloor \frac{N}{d} \rfloor + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

The rest of the proof is similar to that of Cor. 3.3 in [14].

Appendix E
Proof of Corollary 4.5

The proof is similar to that of [14, Cor. 3.4]. To accommodate the alignment set $\mathcal{A}$, one need to change (24) and (25) in [14] to

$$c_{kjpq}^{\mathrm{t}} + c_{kjpq}^{\mathrm{r}} = \begin{cases} 1 & \text{if: } (k,j) \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}, \tag{37}$$

$$\sum_{j=1,\neq k}^{\tilde{K}} \sum_{q=1}^{d_j} c_{kjpq}^{\mathrm{r}} \le N_k^{(\ell)} - d_k, \quad \forall k \in \{1,2,\ldots,K\} \tag{38}$$

respectively. Then the rest of the proof follows.

Appendix F
Proof of Theorem 4.5

The theorem will be proved by contradiction.

Suppose there exists a local optimum $\{\tilde{\mathbf{U}}_k^{\mathrm{L}}, \tilde{\mathbf{V}}_j^{\mathrm{L}}\}$ such that

$$F(\{g_{kjpq}(\tilde{\mathbf{U}}_k^{\mathrm{L}}, \tilde{\mathbf{V}}_j^{\mathrm{L}})\}) > 0. \tag{39}$$

Note that $\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{g_{kjpq}\}) = \mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{f_{kjpq}\})$. Hence, from Theorem 4.1 and 4.3, when IA is feasible, the set $\{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\} : \mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{g_{kjpq}\})$ is full row rank.$\}$ is dense. Therefore, for any $\delta > 0$, there exists a $\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}$ satisfying:

$$\sum_k \|\tilde{\mathbf{U}}_k^{\mathrm{L}} - \tilde{\mathbf{U}}_k\|_{\mathrm{F}} + \sum_j \|\tilde{\mathbf{V}}_j^{\mathrm{L}} - \tilde{\mathbf{V}}_j\|_{\mathrm{F}} \le \delta^2 \tag{40}$$

$$\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{g_{kjpq}\}) \text{ is full row rank.} \tag{41}$$

Since both F and all g_{kjpq} continuously differentiable, $\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(F)$ is finite on any bounded close set. Therefore, from (40), there exists some finite constant $C \ge 0$ such that

$$|F(\{g_{kjpq}(\tilde{\mathbf{U}}_k^{\mathrm{L}}, \tilde{\mathbf{V}}_j^{\mathrm{L}})\}) - F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\})| \le C\delta^2. \tag{42}$$

When a matrix $\mathbf{A}$ is full row rank, the linear equation set $\mathbf{Ax} = \mathbf{b}$ has solution for any vector $\mathbf{b}$. Therefore, from (41), there exists $\{\Delta\tilde{\mathbf{U}}_k, \Delta\tilde{\mathbf{V}}_j\}$ that satisfies linear equation set

$$\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{g_{kjpq}\}) \begin{bmatrix} \mathrm{Vec}\{\Delta\tilde{\mathbf{U}}_1\} \\ \vdots \\ \mathrm{Vec}\{\Delta\tilde{\mathbf{U}}_K\} \\ \mathrm{Vec}\{\Delta\tilde{\mathbf{V}}_1\} \\ \vdots \\ \mathrm{Vec}\{\Delta\tilde{\mathbf{V}}_{\tilde{K}}\} \end{bmatrix} = \begin{bmatrix} g_{1211}(\tilde{\mathbf{U}}_1, \tilde{\mathbf{V}}_2) \\ \vdots \\ g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j) \\ \vdots \\ g_{K\tilde{K}d_K d_{\tilde{K}}}(\tilde{\mathbf{U}}_K, \tilde{\mathbf{V}}_{\tilde{K}}) \end{bmatrix}. \tag{43}$$

From (43), one can obtain (44), where $\|\Delta\mathbf{g}\| \sim \mathcal{O}(\delta^2)$. Denote $\Delta\mathbf{g}$ by $[\Delta g_{1211}, \ldots, \Delta g_{kjpq}, \ldots, \Delta g_{K\tilde{K}d_K d_{\tilde{K}}}]^{\mathrm{T}}$.

$$\begin{aligned} &\begin{bmatrix} g_{1211}(\tilde{\mathbf{U}}_1 - \delta\Delta\tilde{\mathbf{U}}_1, \tilde{\mathbf{V}}_2 - \delta\Delta\tilde{\mathbf{V}}_2) \\ \vdots \\ g_{K\tilde{K}d_K d_{\tilde{K}}}(\tilde{\mathbf{U}}_K - \delta\Delta\tilde{\mathbf{U}}_K, \tilde{\mathbf{V}}_{\tilde{K}} - \delta\Delta\tilde{\mathbf{V}}_{\tilde{K}}) \end{bmatrix} \\ &= \begin{bmatrix} g_{1211}(\tilde{\mathbf{U}}_1, \tilde{\mathbf{V}}_2) \\ \vdots \\ g_{K\tilde{K}d_K d_{\tilde{K}}}(\tilde{\mathbf{U}}_K, \tilde{\mathbf{V}}_{\tilde{K}}) \end{bmatrix} - \delta\mathbf{J}_{\{\tilde{\mathbf{U}}_k,\tilde{\mathbf{V}}_j\}}(\{f_{kjpq}\}) \begin{bmatrix} \mathrm{Vec}\{\Delta\tilde{\mathbf{U}}_1\} \\ \vdots \\ \mathrm{Vec}\{\Delta\tilde{\mathbf{V}}_K\} \end{bmatrix} + \Delta\mathbf{g} \\ &= (1-\delta) \begin{bmatrix} g_{1211}(\tilde{\mathbf{U}}_1, \tilde{\mathbf{V}}_2) \\ \vdots \\ g_{K\tilde{K}d_K d_{\tilde{K}}}(\tilde{\mathbf{U}}_K, \tilde{\mathbf{V}}_{\tilde{K}}) \end{bmatrix} + \Delta\mathbf{g} \end{aligned} \tag{44}$$

Further note that F is convex, continuously differentiable and $F(0,\ldots,0) = 0$. From (44), there exists some constant $\tilde{C} > 0$ such that

$$\begin{aligned} &F(\{g_{kjpq}(\tilde{\mathbf{U}}_k - \delta\Delta\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j - \delta\Delta\tilde{\mathbf{V}}_j)\}) \\ &= F(\{(1-\delta)g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j) + \Delta g_{kjpq}\}) \\ &\le F(\{(1-\delta)g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) + \tilde{C}\delta^2 \\ &\le \delta F(0,\ldots,0) + (1-\delta)F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) + \tilde{C}\delta^2 \\ &= (1-\delta)F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) + \tilde{C}\delta^2. \end{aligned} \tag{45}$$

From (42) and (45),

$$\begin{aligned} &F(\{g_{kjpq}(\tilde{\mathbf{U}}_k^{\mathrm{L}}, \tilde{\mathbf{V}}_j^{\mathrm{L}})\}) \\ &- F(\{g_{kjpq}(\tilde{\mathbf{U}}_k - \delta\Delta\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j - \delta\Delta\tilde{\mathbf{V}}_j)\}) \\ &= (F(\{g_{kjpq}(\tilde{\mathbf{U}}_k^{\mathrm{L}}, \tilde{\mathbf{V}}_j^{\mathrm{L}})\}) - F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\})) \\ &\quad + (F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) \\ &\quad - F(\{g_{kjpq}(\tilde{\mathbf{U}}_k - \delta\Delta\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j - \delta\Delta\tilde{\mathbf{V}}_j)\})) \\ &\ge -C\delta^2 + \delta F(\{g_{kjpq}(\tilde{\mathbf{U}}_k, \tilde{\mathbf{V}}_j)\}) - \tilde{C}\delta^2 \\ &\ge \delta F(\{g_{kjpq}(\tilde{\mathbf{U}}_k^{\mathrm{L}}, \tilde{\mathbf{V}}_j^{\mathrm{L}})\}) - (C+\tilde{C})\delta^2 - C\delta^3. \end{aligned} \tag{46}$$

If (39) is true, when δ is sufficiently small, (46) is positive, which contradicts the assumption that $\{\tilde{\mathbf{U}}_k^{\mathrm{L}}, \tilde{\mathbf{V}}_j^{\mathrm{L}}\}$ is a local optimum. This completes this proof.

Appendix G
Proof of Corollary 4.6

Function $F(\{x_i\}) = \sum_i x_i x_i^\dagger$ is convex and continuously differentiable, with $F(\{0\}) = 0$. Hence, from Theorem 4.5, one only needs to show that the output of Algorithm 1, i.e., $\{\mathbf{V}_j^*, \mathbf{U}_k^{(\ell)^*}\}$, is a local optimum. In Step 2 and 3 of Algorithm 1, the updated $\tilde{\mathbf{U}}_k$, and $\tilde{\mathbf{V}}_j$, given by (20) and (21) are respectively the optimal solutions of the following two sets of unconstraint quadratic optimization problems:

1) Problem G.1 (Interference Optimization at LR k):

$$\underset{\tilde{\mathbf{U}}_k}{\text{minimize}} \quad \sum_{j:(j,k)\in\mathcal{A}} \left\| \begin{bmatrix} \mathbf{I}_{d_k \times d_k} \\ \tilde{\mathbf{U}}_k \end{bmatrix}^\dagger \mathbf{H}_{kj}\mathbf{V}_j \right\|_{\mathrm{F}}^2. \tag{47}$$

2) Problem G.2 (Interference Optimization at LT j):

$$\underset{\tilde{\mathbf{V}}_j}{\text{minimize}} \quad \sum_{k:(j,k)\in\mathcal{A}} \left\| \mathbf{U}_k^\dagger \mathbf{H}_{kj} \begin{bmatrix} \mathbf{I}_{d_j \times d_j} \\ \tilde{\mathbf{V}}_j \end{bmatrix} \right\|_{\mathrm{F}}^2. \tag{48}$$

Therefore, $\sum_{k=1}^K \sum_{j:(j,k)\in\mathcal{A}} \|\mathbf{U}_k^\dagger \mathbf{H}_{kj}\mathbf{V}_j\|_{\mathrm{F}}^2$ is non-increasing in every round of update. Further noting that $\sum_{k=1}^K \sum_{j:(j,k)\in\mathcal{A}} \|\mathbf{U}_k^\dagger \mathbf{H}_{kj}\mathbf{V}_j\|_{\mathrm{F}}^2 \ge 0$, Algorithm 1 must converge to a local optimum. This completes the proof.

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REFERENCES

[1] I. Sason, "On achievable rate regions for the Gaussian interference channel," *IEEE Trans. Inf. Theory*, vol. 50, no. 6, pp. 1345–1356, Jun. 2004.

[2] R. H. Etkin, D. N. C. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 1345–1356, Dec. 2008.

[3] M. A. Maddah-ali, A. S. Motahari, and A. K. Khandani, "Communication over MIMO X channels: Interference alignment, decomposition, performance analysis," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3457–3470, Aug. 2008.

[4] V. Cadambe and S. Jafar, "Interference alignment and degrees of freedom of the K-user interference channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[5] M. Z. Win, A. Rabbachin, J. Lee, and A. Conti, "Cognitive network secrecy with interference engineering," *IEEE Netw.*, vol. 28, no. 5, pp. 86–90, Sep.–Oct. 2014.

[6] M. Z. Win, L. Ruan, A. Rabbachin, Y. Shen, and A. Conti, "Multi-tier network secrecy in the ether," *IEEE Commun. Mag.*, vol. 53, no. 6, pp. 28–32, Jun. 2015.

[7] O. Koyluoglu, H. El Gamal, L. Lai, and H. Poor, "Interference alignment for secrecy," *IEEE Trans. Inf. Theory*, vol. 57, no. 6, pp. 3323–3332, Jun. 2011.

[8] R. Bassily and S. Ulukus, "Ergodic secret alignment," *IEEE Trans. Inf. Theory*, vol. 58, no. 3, pp. 1594–1611, Mar. 2012.

[9] C. M. Yetis, T. Gou, S. A. Jafar, and A. H. Kayran, "On feasibility of interference alignment in MIMO interference networks," *IEEE Trans. Signal Process.*, vol. 58, pp. 4771–4782, Sep. 2010.

[10] G. Bresler, D. Cartwright, and D. Tse, "Settling the feasibility of interference alignment for the MIMO interference channel: The symmetric square case," in *Proc. IEEE Inf. Theory Workshop*, Paraty, Brazil, Oct. 2011, pp. 447–451.

[11] M. Razaviyayn, L. Gennady, and Z. Luo, "On the degrees of freedom achievable through interference alignment in a MIMO interference channel," *IEEE Trans. Signal Process.*, vol. 60, no. 2, pp. 812–821, Feb. 2012.

[12] C. Wang, T. Gou, and S. A. Jafar, "Subspace alignment chains and the degrees of freedom of the three-user MIMO interference channel," in *Proc. IEEE Int. Symp. Inf. Theory*, Cambridge, MA, USA, Jul. 2012, pp. 2471–2475.

[13] O. Gonález, C. Beltrán, and I. Santamaría, "A feasibility test for linear interference alignment in MIMO channels with constant coefficients," *IEEE Trans. Inf. Theory*, vol. 60, no. 3, pp. 1840–1856, Mar. 2014.

[14] L. Ruan, V. K. Lau, and M. Z. Win, "The feasibility conditions for interference alignment in MIMO networks," *IEEE Trans. Signal Process.*, vol. 61, no. 8, pp. 2066–2077, Apr. 2013.

[15] N. Lee, D. Park, and Y. Kim, "Degrees of freedom on the K-user MIMO interference channel with constant channel coefficients for downlink communications," in *Proc. IEEE Global Telecommun. Conf.*, Honolulu, HI, USA, Dec. 2009, pp. 1–6.

[16] R. Tresch, M. Guillaud, and E. Riegler, "On the achievability of interference alignment in the K-user constant mimo interference channel," in *Proc. IEEE/SP 15th Workshop Statist. Signal Process.*, Cardiff, U.K., Aug. 2009, pp. 277–280.

[17] M. Khatiwada and S. W. Choi, "On the interference management for k-user partially connected fading interference channels," *IEEE Trans. Commun.*, vol. 60, no. 12, pp. 3717–3725, Dec. 2012.

[18] K. Gomadam, V. R. Cadambe, and S. A. Jafar, "Approaching the capacity of wireless networks through distributed interference alignment," in *Proc. IEEE Global Telecommun. Conf.*, New Orleans, LA, USA, Nov. 2008, pp. 1–6.

[19] S. W. Peters and R. W. Heath, Jr., "Cooperative algorithms for MIMO interference channels," *IEEE Trans. Veh. Technol.*, vol. 60, no. 1, pp. 206–218, Jan. 2011.

[20] K. Gomadam, V. R. Cadambe, and S. A. Jafar, "A distributed numerical approach to interference alignment and applications to wireless interference networks," *IEEE Trans. Inf. Theory*, vol. 57, no. 6, pp. 3309–3322, Jun. 2011.

[21] D. S. Papailiopoulos and A. G. Dimakis, "Interference alignment as a rank constrained rank minimization," *IEEE Trans. Signal Process.*, vol. 60, no. 8, pp. 4278–4288, Aug. 2012.

[22] I. Shafarevich, *Basic Algebraic Geometry: Volume I and II*, 2nd ed. New York, NY, USA: Springer, 1996.

[23] D. Cox, J. Little, and D. O'Shea, *Ideals Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 3rd ed. New York, NY, USA: Springer, 2006.

[24] J. Harris, *Algebraic Geometry: A First Course*, 1st ed. New York, NY, USA: Springer, 1992.

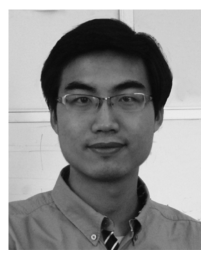
[25] M. Razaviyayn, M. Sanjabi, and Z. Luo, "Linear transceiver design for interference alignment: Complexity and computation," *IEEE Trans. Inf. Theory*, vol. 58, no. 5, pp. 2896–2910, May 2012.

[26] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, 1st ed. Philadelphia, PA, USA: SIAM, 2000.

[27] G. Kemper, *A Course in Commutative Algebra*, 1st ed. New York, NY, USA: Springer, 2010.

[28] R. Ehrenborg and G.-C. Rota, "Apolarity and canonical forms for homogeneous polynomials," *Eur. J. Combin.*, vol. 14, no. 3, pp. 157–181, May 1993.

[29] L.-H. Hsu and C.-K. Lin, *Graph Theory and Interconnection Networks*. Boca Raton, FL, USA: CRC Press, 2009.



Liangzhong Ruan (S'10–M'14) received the Ph.D. degree in electrical engineering and computer science from the Hong Kong University of Science and Technology (HKUST), Clear Water Bay, in 2013, and the B.Eng. degree in electrical engineering from Tsinghua University, Beijing, China, in 2007.

From 2012 to 2013, he was with Massachusetts Institute of Technology (MIT), Cambridge, USA, as a visiting graduate student. He is currently a postdoctoral associate at MIT. His research interests include interference management, intrinsic wireless secrecy, and quantum entanglement distillation.

Vincent K. N. Lau (M'98–SM'01–F'11) received the B.Eng. degree from the University of Hong Kong in 1992 and the Ph.D. degree from Cambridge University, U.K., in 1997.

He joined the Hong Kong University of Science and Technology (HKUST), Clear Water Bay, in 2004 and was promoted to Professor in 2010. His research interests include the delay-sensitive optimization, interference management, and cooperative communications. He is the founder and director of Huawei-HKUST Innovation Lab.

Moe Z. Win (S'85–M'87–SM'97–F'04) is a professor at the Massachusetts Institute of Technology (MIT), Cambridge, MA, USA. Prior to joining MIT, he was with AT&T Research Laboratories for five years and with the Jet Propulsion Laboratory for seven years. His current research topics include network localization and navigation, network interference exploitation, intrinsic wireless secrecy, adaptive diversity techniques, ultra-wide bandwidth systems, and quantum information science. He is an elected Fellow of the AAAS, the IEEE, and the IET, and was an IEEE Distinguished Lecturer.